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COMPANION TO ALGEBRA

WITH NUMEROUS EXAMPLES

BY

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ASSISTANT MASTER AT CHARTERHOUSE

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PREFACE.

THIS work is intended especially for those candidates preparing for the Woolwich Entrance and similar examinations, who are already familiar with the easier parts of Algebra, as given in any of the elementary text-books, but are not equal to attacking the larger and more systematic treatises. I have endeavoured to make the selection of Theorems and Examples as interesting and useful as possible, and I have ventured, on account of the importance of the results, to give one or two proofs which are not as logically complete as might be wished. Those of the examples which are not original I have obtained from Army Entrance, University and School Examination Papers.

L. MARSHALL.

CHARTERHOUSE, GODALMING, 1882.

CONTENTS.

COMPANION TO ALGEBRA.

I.—ELEMENTARY FORMULÆ AND RESULTS.

THE following formulæ in §§ 1 and 3-9 can be directly verified by multiplication, and should be committed to memory.

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

ERRATA.

Page 40, bottom line, *for* $x^{\frac{pr}{q}}$ *read* $x^{\frac{pr}{qs}}$.

Page 47, Ex. 17, *for* $\frac{1}{4}x^{\frac{1}{2}}a^{\frac{1}{2}}$ *read* $\frac{1}{4}x^{\frac{1}{2}}a^{-\frac{1}{2}}$.

Page 58, Ex. 56, *for* 128 *read* 28.

Page 87, Ex. 20, *for* x^n *read* x.

Page 102, 2nd line from bottom, *for* $F(x)$ *read* $f(x)$.

Page 143, Ex. 177, *read*

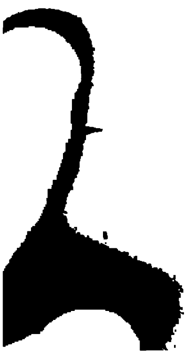
$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots$$

$$e.g. \left(a - 2b + \frac{c^3}{2} - d^2\right)^2 = a^2 + 4b^2 + \frac{c^6}{4} + d^4 + 2a(-2b) + 2a \cdot \frac{c^3}{2} + 2a(-d^2)$$

$$- 4b \cdot \frac{c^3}{2} - 4b(-d^2) + c^3(-d^2)$$

$$= a^2 + 4b^2 + \frac{c^6}{4} + d^4 - 4ab + ac^3 - 2ad^2 - 2bc^3 + 4bd^2 - c^3d^2.$$

3. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.



COMPANION TO ALGEBRA.

I.—ELEMENTARY FORMULÆ AND RESULTS.

The following formulæ in §§ 1 and 3-9 can be directly verified by multiplication, and should be committed to memory.

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

e.g. $(2ax - 3y^2)^2 = (2ax)^2 - 2 \, . \, 2ax \, . \, 3y^2 + (3y^2)^2 = 4a^2x^2 - 12axy^2 + 9y^4$.

Notice that $(a-b)^2 = (b-a)^2$.

2. $(a+b+c+d+ \dots + k)^2 =$ *sum of squares of* $a, b, \dots k +$ *twice the product of each pair, formed by taking each term in turn with each of those that follow it.*

Proof. $(a+b+c+d+ \dots +k)^2 = (a+\overline{b+c+ \dots +k})^2$

$$= a^2 + 2a(b+c+ \dots +k) + (b+\overline{c+ \dots +k})^2$$

$$= a^2 + 2a(b+ \dots +k) + b^2 + 2b(c+ \dots +k) + (c+\overline{d+ \dots k})^2,$$

and so on till we get the squares of all the letters.

It will be seen that this result may also be arranged in the following way:—

$$(a+b+c+d+ \dots +k)^2 = a^2 + (2a+b)b + (2a+2b+c)c + \dots$$
$$\dots + (2a+2b+2c+ \dots +k)k.$$

$$e.g.\left(a - 2b + \frac{c^3}{2} - d^2\right)^2 = a^2 + 4b^2 + \frac{c^6}{4} + d^4 + 2a(-2b) + 2a \cdot \frac{c^3}{2} + 2a(-d^2)$$
$$- 4b \cdot \frac{c^3}{2} - 4b(-d^2) + c^3(-d^2)$$
$$= a^2 + 4b^2 + \frac{c^6}{4} + d^4 - 4ab + ac^3 - 2ad^2 - 2bc^3 + 4bd^2 - c^3d^2.$$

3. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

4. $(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$.

5. $(a+b)(a-b) = a^2 - b^2$, *i.e. the product of the sum and difference of two quantities = the difference of their squares.*

$$e.g.\ (127)^2 - (123)^2 = (127+123)(127-123) = 250 \times 4 = 1000.$$
$$9c^2 - 4(a-b)^2 = (3c+2a-2b)(3c-2a+2b).$$
$$(a-b+c+d)(a+b-c+d) = \{(a+d)-(b-c)\}\{(a+d)+(b-c)\}$$
$$= (a+d)^2 - (b-c)^2 = a^2 + 2ad + d^2 - b^2 + 2bc - c^2.$$

6. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

$$e.g.\ 8a^3 + \frac{b^6}{27} = (2a)^3 + \left(\frac{b^2}{3}\right)^3 = \left(2a + \frac{b^2}{3}\right)\left(4a^2 - \frac{2ab^2}{3} + \frac{b^4}{9}\right).$$

7. $a^4 + a^2b^2 + b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$.

$$e.g.\ x^8 + x^4 + 1 = (x^4+x^2+1)\ (x^4-x^2+1)$$
$$= (x^2+x+1)\ (x^2-x+1)(x^4-x^2+1).$$

8. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

9. $(x+a)(x+b) = x^2 + (a+b)x + ab$.

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$$
$$(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3$$
$$+ (ab+ac+ad+bc+bd+cd)x^2 + (bcd+acd+abd+abc)x + abcd.$$

Notice that the co-efficient of x in the last formula is obtained by leaving out each letter in turn.

$$e.g.\ \left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = x^2 + \frac{5}{6}x + \frac{1}{6};\ (x^2y^2 - xy - 12) = (xy-4)(xy+3);$$
$$(x+1)(x+2)(x-3) = x^3 + (1+2-3)x^2 + (2-3-6)x - 6 = x^3 - 7x - 6.$$

10. $\dfrac{x^n+y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ \ldots\ + y^{n-1}$, when n is *odd.*

$\dfrac{x^n-y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ \ldots\ - y^{n-1}$, when n is *even.*

$\dfrac{x^n-y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}$, when n is either *odd* or *even.*

The last three formulæ may be stated otherwise, and proved by induction thus :—

e.g. $x^n + y^n$ is divisible by $x+y$ when n is *odd.*

For $\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+\frac{x^{n-2}+y^{n-2}}{x+y}\cdot y^2$ by actual division; therefore, if $x^{n-2}+y^{n-2}$ is divisible by $x+y$, so also is x^n+y^n, but we know (§ 6) that x^3+y^3 is divisible by $x+y$, therefore x^5+y^5 is also, and so by induction is x^n+y^n when n is any *odd* integer.

These results are most easily remembered by the simple cases of §§ 5, 6.

11. The following important results are proved on the supposition that the quantities a, b, c, d are all positive, and that $a>b$, $b>c$, $c>d$.

(A.) To prove $\left.\begin{array}{l}\text{(i.) } a+(b-c)=a+b-c \\ \text{and (ii.) } a-(b-c)=a-b+c\end{array}\right\}$

(i.) To get $a+(b-c)$, we have to add $(b-c)$ to a. Now, if we add b to a, we get $a+b$, but we should have added c too much. Therefore for the true result we must subtract c; we thus get $a+(b-c)=a+b-c$.

(ii.) To get $a-(b-c)$, we have to subtract $(b-c)$ from a. Now, if we subtract b from a, we get $a-b$, but we should have subtracted c too much. Therefore we must add c to the result: we thus get $a-(b-c)=a-b+c$.

Similarly, $\left.\begin{array}{l}a+(b+c)=a+b+c \\ \text{and } a-(b+c)=a-b-c\end{array}\right\}$

(B.) To prove the *Law of Signs* in multiplication, *i.e.* that

$$(a-b)(c-d)=ac-bc-ad+bd.$$

Put $a-b=M$, then M is a *positive* quantity, $\because a>b$ by supposition.

$$\therefore (a-b)(c-d)=M(c-d).$$

Now, to get M multiplied by $(c-d)$, if we take Mc, that is M multiplied by c, we are multiplying by c instead of $c-d$, that is by d times too much; therefore, to get the true result, we must subtract M multiplied by d, *i.e.* dM.

$$\begin{aligned}\text{Therefore } M(c-d)&=cM-dM\\ \therefore (a-b)(c-d)&=c(a-b)-d(a-b)\\ &=(ac-bc)-(ad-bd) \text{ as before}\\ &=ac-bc-ad+bd \text{ by (A).}\end{aligned}$$

EXAMPLES. I.

1. Prove that $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2+2abc(a+b+c)$
$=2(bc+ca+ab)^2$.

2. Find the cube of $a+b+c$, and deduce that of $a-b-c$.

3. Prove that
$(x+z)^3+3(x+z)^2y+3(x+z)y^2+y^3=(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$.

4. Divide $9a^2+6ab+b^2-4c^2-4cd-d^2$ by $3a+b-2c-d$.

5. If $s=\dfrac{a+b+c}{2}$, prove that $16s(s-a)(s-b)(s-c)=(a+b+c)(b+c-a)$ $(c+a-b)(a+b-c)=2(b^2c^2+c^2a^2+a^2b^2)-a^4-b^4-c^4$.

6. Find the continued product of x^2+x+1, x^2+x-1, and $x^4-2x^3+x^2+1$.

7. Find two factors of $x^{3n}-y^{3m}$ where m, n are positive integers.

8. Divide $(x^2-yz)^3+8y^3z^3$ by x^2+yz.

9. Show that
$$\frac{(a+b)^3-c^3}{a+b-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b}=2(a+b+c)^2+a^2+b^2+c^2.$$

10. Divide $\left(x^2+\dfrac{1}{x^2}\right)^3-8$ by $\left(x-\dfrac{1}{x}\right)^2$.

11. Resolve $x^2-\left(a-\dfrac{1}{a}\right)x-1$ into two factors, and
$(x^2-3x)^2-2(x^2-3x)-8$ into four factors.

12. Prove that $\dfrac{x^5+a^5}{x+a}=a^4-x[a^3-x\{a^2-x(a-x)\}]$.

13. Write down the quotient of $16-81a^4$ divided by $2-3a$.

14. Simplify $(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)$.

15. Find two factors of $x^3-y^3+1+3xy$.

16. Prove that $a(a+1)(a+2)(a+3)+1=(a^2+3a+1)^2$.

17. Divide $1-5x+\dfrac{152}{15}x^3-\dfrac{106}{225}x^4-\dfrac{28}{9}x^5$ by $1-x-\dfrac{14}{15}x^2$.

18. Multiply together $(x+a-b)(x+b-c)(x+c-a)$, collecting in brackets the co-efficients of the powers of x. If $a-b=b-c$, express the result in terms of x, a and b only.

19. Simplify $a(b+c)^2+b(c+a)^2+c(a+b)^2+(a-b)(b+c)(a-c)$
$+(b-c)(c+a)(b-a)+(c-a)(a+b)(c-b)$.

20. Divide $(x+y)^2+(z+x)^2+(y+z)^2+2(x+y)(x+z)+2(y+z)(y+z)$
$+2(z+x)(z+y)$ by $x+y+z$.

21. Multiply
$(1+x+2x^2)^2-(1-x-2x^2)^2$ by $(1+x-2x^2)^2-(1-x+2x^2)^2$.

22. Divide $x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2$ by $x^2-(a+b)x+ab$.

23. If $x=b+c-a$, $y=c+a-b$, and $z=a+b-c$, find the value of
$x^2+y^2+z^2+2yz+2zx+2xy$.

24. Divide $(ac+bd)^2-(ad+bc)^2$ by $(a-b)(c-d)$.

25. If $2s=a+b+c$, prove that $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

26. If $A=a^2+b^2+c^2$, and $B=bc+ca+ab$, prove that $A^3+2B^3-3AB^2$
$=(a^3+b^3+c^3-3abc)^2$.

27. Express $a^2(c-b)+b^2(a-c)+c^2(b-a)$ as the product of three simple factors.

28. Divide $(m+1)(bx+an)b^2x^2-(n+1)(mbx+a)a^2$ by $bx-a$.

29. Find the product, and the sum of the squares of the expressions
$ax+by$, $bx-ay$, $ay+bx$, and $by-ax$.

30. Multiply $\{(x^2+x+1)a-(x+1)\}$ by $\{(x-1)a^2-(x-1)a+3\}$.

31. Divide $(x^3-3x^2y)^2-(3xy^2-y^3)^2$ by $(x-y)^3$.

32. Find the simple trinomial factors of $(x^2-y^2-z^2)^2-4y^2z^2$.

33. Find the quotient when $x^4-(b-a-c+d)x^3+(ac-ab-ad-bc+bd-cd)x^2+(abd-abc-acd+bcd)x+abcd$ is divided by the continued product of $x+a$, $x-b$, and $x+c$.

34. Find the product of $m-n-p-q$, and $m+n+p+q$.

35. Find the factors of $x^2-2mx+(m^2-n^2)$, and of
$(x-y)(x^2-z^2)-(x-z)(x^2-y^2)$.

36. Find the continued product of—
$x+3y$, $x-5y$, and $x+4y$;
also of $x+1$, $x-2$, $x-3$, and $x+4$.

37. Multiply together $x+\frac{1}{2}$, $x-\frac{1}{3}$ and $x-\frac{1}{6}$.

38. Resolve $x^8-\frac{1}{256}$ as far as possible into rational factors.

39. Divide $2a^2x^2 - 2(3b-4c)(b-c)y^2 + abxy$ by $ax + 2(b-c)y$.

40. Simplify $(b-c)(x+b)(x+c) + (c-a)(x+c)(x+a)$
$$+ (a-b)(x+a)(x+b).$$

41. Simplify $a(b-c)(b+c-a)^2 + b(c-a)(c+a-b)^2 + c(a-b)(a+b-c)^2$.

42. If $x+y+z-1 = \sqrt{2(1-x)(1-y)(1-z)}$,
prove that $x^2+y^2+z^2-1+2xyz=0$.

43. If $(b+c)x=a$, $(c+a)y=b$, $(a+b)z=c$,
prove that $yz+zx+xy+2xyz=1$.

44. Multiply $(x+y+2\sqrt{x+y}+4)$ by $(x+y-2\sqrt{x+y}+4)$.

45. Prove $$\frac{2n(2n-1)(2n-2)}{1.2.3} - \frac{(2n)^2(2n-1)}{1.2} + \frac{(2n)^2(2n+1)}{1.2}$$
$$-\frac{2n(2n+1)(2n+2)}{1.2.3} = 0.$$

46. Find the difference between the squares of 3503 and 3497.

47. Simplify the expression
$$16x - 10 - [7 - \{8x - (9x - \overline{3 - 6x})\}],$$
and find the least integral value of x which will make the result negative.

48. Multiply the square of the sum of the cubes of a and b by the cube of the sum of their squares.

49. If $(a+bc)^2(1-a^2) = (b+ac)^2(1-b^2)$, prove that $a^2+b^2+c^2+2abc=1$.

50. Find the algebraical expression which, when divided by x^2+x-1, gives x^3-3x^2+4x-7 for the quotient and $11x-7$ for the remainder.

51. Find the continued product of $x+1$, $x+3$, $x-2$, and $x-4$.

52. Find $(2a+3b-\frac{c}{2}-\frac{d}{4}+e)^2$ and $(a+b+c+d)^3$.

53. If $2s=a+b+c$, express $(s-a)^2+(s-b)^2+(s-c)^2+s^2$ in terms of a, b, c.

54. Divide $4a(a+b+c)+10bc-3(b^2+c^2)$ by $2a+3b-c$.

55. Write down the quotients of x^8-y^8 when divided by $x+y$ and $x-y$; and of x^5+y^5 when divided by $x+y$.

56. Simplify $a[b-c\{d-e(f-g.\overline{h-k})\}]$.

57. Find the difference between the squares of 2049 and 2051.

58. If $s=\frac{a+b+c}{2}$, prove that $(b-c)(s-a)^2 = s^2(b-c) - a(b^2-c^2)$.

59. Simplify $x^3+(x+y)^3+(x-y)^3-3x(x^2-y^2)$.

60. From $\{m(2m-3p)-2n(4n-3p)\}x+\{m(p-m)-p(2n+p)\}y$

take $3\left\{p\left(2n-\frac{3p}{2}\right)-\frac{p}{2}(2m-3p)\right\}x-\{p(p-m)+2n(2n+p)\}y$;

and find the value of the difference when $x=n=\frac{1}{2}$, $y=m=-2$.

II.—G.C.M. L.C.M., ETC.

1. *Definitions.*—One number or quantity (P) is said to *measure* another (A) when it *divides* it without remainder, *i.e.* is a *factor* of it, so that $A=aP$. In Arithmetic A, a, P, are necessarily whole numbers. The greatest number or highest quantity which measures two or more numbers or quantities is called their *Greatest Common Measure* (G.C.M.), their *Highest Common Divisor* (H.C.D.), or their *Highest Common Factor* (H.C.F.)

Again, if $A=aP$, A is said to be a *multiple* of P. The least number or lowest quantity which contains two or more others as factors is called their *Least Common Multiple* (L.C.M.)

2. Let P measure both A and B, then by (1) $A=kP$ and $B=lP$. Therefore $mA=(mk)P$; $\therefore$ P measures mA. And $mA\pm nB=mkP\pm nlP=(mk\pm nl)P$; $\therefore$ P measures $mA\pm nB$.

Therefore (i.) if one quantity measures another, it also measures any multiple of it; and (ii.) if one quantity measures both of two others, it also measures the sum or difference of any multiples of them.

3. *To prove the rule for finding the G.C.M. of any two expressions A and B, showing how factors may be introduced or rejected in the process.*

Arrange A, B according to descending powers of any common letter. If necessary multiply one of them (say A) by a, which must not be contained in B, so that the product Aa when divided by B may give a

quotient with an integral co-efficient. Divide as shown, and if possible at any step divide out the remainder by any factor which is not contained in both A and B. The last remainder D will be the required G.C.M. of A and B.

$$
\begin{array}{l}
B)Aa(p \\
\quad \underline{pB} \\
\quad c)C \\
\qquad C')B(q \\
\qquad\quad \underline{qC'} \\
\qquad\quad D)C'(r \\
\qquad\qquad \underline{rD}
\end{array}
$$

Then $\because C'=rD$, $\therefore D$ measures C',
$\therefore D$ measures $D+qC'$, *i.e.* B [*see* § 2];
$\therefore D$ measures $cC'+pB=C+pB=aA$;
$\therefore D$ is a C.M. of aA and B, and $\therefore$ of A and B;
$\because a$ contains no measure of B.
Next, whatever measures A and B must measure $aA-pB$, *i.e.* C, and $\therefore$ must measure C', for c contains no C.M. of A and B. Therefore whatever measures A and B must measure $B-qC'$, *i.e.* D; but nothing greater than D can measure D, $\therefore D$ is the G.C.M. of A and B. In the same way as a is introduced at the first step, so any remainder may be multiplied by a quantity which is not contained in either A or B.

If we *divide* or *multiply both* A and B by the same quantity (p) so as to get A' and B', then from first principles it is clear that the G.C.M. of A and B is found by *multiplying* or *dividing* that of A' and B' by p.

4. If we have simply to prove the rule for finding the G.C.M. of A and B when no divisions or multiplications as above are made during the process, the proof will be of the same kind and the division as follows :—

$$
\begin{array}{l}
B)A(p \\
\quad \underline{pB} \\
\quad C)B(q \\
\qquad \underline{qC} \\
\qquad D)C(r \\
\qquad\quad \underline{rD}
\end{array}
$$

First we prove that D is a C.M. of A and B, and then that it is the G.C.M.

5. *To find the G.C.M. of more than two quantities.*

Let A, B, C be any three quantities. Let g be the G.C.M. of A and B. Then g contains the highest factor common to both A and B; and if G be the G.C.M. of g and C, it will clearly contain the highest factor common to both g and C, and therefore to all three A, B, and C. Therefore G is the G.C.M. of A, B, and C.

The same rule can clearly be extended to any number of quantities.

6. In finding the G.C.M. of two expressions which begin with high numbers, it is often convenient to get rid of the terms at the right-hand end first. Thus to find the G.C.M. of $126x^2-55x-25$, and $162x^2+63x+5$.

$$162x^2 + 63x + 5$$
$$5$$
$$126x^2 - 55x - 25)\overline{810x^2 + 315x + 25}(-1$$
$$\underline{-126x^2 + 55x + 25}$$
$$4x)\overline{936x^2 + 260x}$$
$$13)\overline{234x + 65}$$
$$18x + 5)126x^2 - 55x - 25(7x - 5$$
$$\underline{126x^2 + 35x}$$
$$-90x - 25$$
$$\underline{-90x - 25}$$

7. It is often advisable to get rid of neither first nor last terms, but to get a remainder of a simpler form though not of a lower degree.

e.g. To find the G.C.M. of $17x^3 - 33x^2 + x - 6$, and $18x^3 - 35x^2 - 4$.

$$17x^3 - 33x^2 + x - 6)18x^3 - 35x^2 - 4 \quad (1$$
$$\underline{17x^3 - 33x^2 + x - 6}$$
$$x^3 - 2x^2 - x + 2)17x^3 - 33x^2 + x - 6(17$$
$$\underline{17x^3 - 34x^2 + 17x + 34}$$
$$x^2 + 18x - 40$$

etc. etc.

and in the usual way the G.C.M. is found to be $x - 2$.

8. Let G and L be the G.C.M. and L.C.M. respectively of A and B. Then $A = aG$, $B = bG$ where a and b have no C.M. Then $\because L$ must be divisible by both A and B,

$\therefore L = abG = aB = bA = \dfrac{AB}{G}$; and $\therefore LG = AB$, *i.e.* the product of any two quantities = the product of the L.C.M. and G.C.M. of the same.

9. If we can resolve A and B into factors, their G.C.M. and L.C.M. may be written down at once.

e.g. Let $A = x^4 - 1 = (x-1)(x+1)(x^2+1)$,

and $B = x^2 - 5x + 4 = (x-1)(x-4)$,

then $G = x - 1$, and $L = (x^4 - 1)(x - 4) = x^5 - 4x^4 - x + 4$.

The same method will apply for finding the G.C.M. or L.C.M. of more than two expressions, if we can resolve them into their factors.

10. Conditions for a *divisor*.

Ex. i. Find the condition that $x^3-3b^2x+2c^3$ may be divisible by $x-a$, whatever may be the value of x.

Divide out and equate the remainder to zero.

$$\begin{array}{l}
x-a)x^3-3b^2x+2c^3(x^2+ax+a^2-3b^2 \\
\underline{x^3-ax^2} \\
\quad ax^2-3b^2x \\
\quad \underline{ax^2-a^2x} \\
\qquad (a^2-3b^2)x+2c^3 \\
\qquad \underline{(a^2-3b^2)x-a^3+3ab^2} \\
\qquad\qquad a^3-3ab^2+2c^3
\end{array}$$

$\therefore$ the required condition is that $a^3-3ab^2+2c^3=0$.

Ex. ii. Find the conditions that $x^3-3b^2x+2c^3$ may be divisible by both $x-a$ and $x-b$.

Proceeding as before we get the two conditions $a^3-3ab^2+2c^3=0$, and $2b^3-2c^3=0$, which are satisfied by $a=b=c$, or $-a=2b=2c$.

Or thus, if we divide out by $(x-a)(x-b)$, *i.e.* $x^2-(a+b)x+ab$

$$\begin{array}{l}
x^2-(a+b)x+ab)x^3-3b^2x+2c^3(x+a+b \\
\qquad \underline{x^3-(a+b)x^2+abx} \\
\qquad (a+b)x^2-(3b^2+ab)x+2c^3 \\
\qquad \underline{(a+b)x^2-(a^2+2ab+b^2)x+a^2b+ab^2} \\
\qquad\quad (a^2+ab-2b^2)x+2c^3-a^2b-ab^2
\end{array}$$

$\therefore (a^2+ab-2b^2)x+2c^3-a^2b-ab^2=0$ for all values of x,

$\therefore a^2+ab-2b^2=0$, and $2c^3-a^2b-ab^2=0$,

which are also satisfied by $a=b=c$, or $-a=2b=2c$.

[These last two examples may be worked by means of § 2, p. 112.]

11. *If* $\dfrac{a}{b}=\dfrac{c}{d}$ *where* a, b, c, d *are all integers, and* c, d *have no common factor, so that the fraction* $\dfrac{c}{d}$ *is in its lowest terms: then* $a=nc$, $b=nd$ *when* n *is an integer.*

For $\because \frac{a}{b}=\frac{c}{d} \therefore a=\frac{bc}{d} \therefore \frac{bc}{d}$ is an integer $\therefore d$ must divide bc; but by supposition d and c have no common factor; $\therefore d$ must divide b, and $\therefore b=nd$, and consequently $a=nc$ where n is some integer.

EXAMPLES. II.

1. G.C.M. of x^3-8x+3 and x^6+3x^5+x+3.
2. G.C.M. of x^4-a^4 and $x^4+a^3x-ax^3-a^4$.
3. G.C.M. of $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.
4. G.C.M. of $x^4-2p(p-q)x^2+(p^2+q^2)(p-q)x-p^2q^2$ and $x^4-(p-q)x^3+(p-q)q^2x-q^4$.
5. G.C.M. of $a(a-1)x^2+(2a^2-1)x+a(a+1)$ and $(a^2-3a+2)x^2+(2a^2-4a+1)x+a(a-1)$.
6. G.C.M. of $3x^3-22x-15$ and $5x^4-17x^3+18x$.
7. G.C.M. of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.
8. G.C.M. of $x^3-12x^2-50x+75$ and $x^3-16x^2+18x-45$.
9. G.C.M. $7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ and $8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$.
10. G.C.M. of $3x^4-4x^3+1$ and $4x^4-5x^3-x^2+x+1$.
11. G.C.M. of $4x^4-13x^2y^2+9y^4$ and $10x^4+23x^3y-28xy^3-15y^4$.
12. G.C.M. of $6x^3+7x^2y-22xy^2-5y^3$ and $15x^3-14x^2y-13xy^2-2y^3$.
13. L.C.M. of x^3+8 and x^4-16.
14. L.C.M. of a^2+b^2, a^2-b^2, $a^3+a^2b+ab^2+b^3$ and $a^3-a^2b+ab^2-b^3$.
15. Express by four trinomial factors the L.C.M. of $(x+2y)^2-9z^2$, $(x+3z)^2-4y^2$, and $(2y+3z)^2-x^2$.
16. Find the L.C.M. of $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ and $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$.
17. L.C.M. of $2x-1$, $6x^2-x-1$, and $8x^2-10x+3$.

18. L.C.M. of $x^2-6ax+9a^2$, $x^2-ax-6a^2$, and $3x^2-12a^2$.

19. L.C.M. of $21x(xy-y^2)^2$, $35(x^4y^2-x^2y^4)$, and $15y(x^2+xy)^2$.

20. G.C.M. of $8x^5-55x^3+3$, and $3x^5-55x^2+8$.

21. G.C.M. of $17x^3+2x^2y+2xy^2-15y^3$ and $15x^3-2x^2y-2xy^2-17y^3$.

22. G.C.M. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

Reduce to their lowest terms the following fractions :—

23. $\dfrac{3x^2-(4a+2b)x+a^2+2ab}{x^3-(2a+b)x^2+(a^2+2ab)x-a^2b}$.

24. $\dfrac{6a^2-13ax+6x^2}{10a^2-9ax-9x^2}$; and $\dfrac{3x^2+8x-3}{3x^2-10x+3}$.

25. $\dfrac{(a^2-b^2)(x^2-y^2)-4abxy}{(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy}$.

26. $\dfrac{x^3y^3-z^6}{(xy-z^2)^2}$; and $\dfrac{x^3y^3+z^3}{x^5y^5+z^5}$.

27. $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}$.

28. $\dfrac{12x^4-4ax^3-23a^2x^2+9a^3x-9a^4}{8x^4-14a^2x^2-9a^4}$.

29. Prove that $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=1$ when $\dfrac{ab-cd}{(a+b)-(c+d)}$ is substituted for x.

30. Subtract $\dfrac{x+3}{x^2+x-12}$ from $\dfrac{x+4}{x^2-x-12}$, and divide the result by

$$1+\frac{2(x^2-12)}{x^2+7x+12}.$$

31. Simplify $\left[\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a^3+x^3}{a^3-x^3}\right]\div\left[\dfrac{a^3-x^3}{a^3+x^3}-\dfrac{a-x}{a+x}\right]$.

32. If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$, prove that

$$(x-a)^2-(y-b)^2=b^2.$$

33. Prove that $x(y+2)+\dfrac{x}{y}+\dfrac{y}{x}$ is equal to a, if $x=\dfrac{y}{y+1}$ and $y=\dfrac{a-2}{2}$.

34. Reduce to its simplest form

$$\frac{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)}{x(z-x)+y(x-y)+z(y-z)}.$$

35. Prove that $\dfrac{a}{(c-a)(a-b)(x+a)}+\dfrac{b}{(a-b)(b-c)(x+b)}$

$$+\frac{c}{(b-c)(c-a)(x+c)}=\frac{x}{(x+a)(x+b)(x+c)}.$$

36. Express $1-\left(\dfrac{a^2+b^2-c^2}{2ab}\right)^2$ as a single fraction with a numerator composed of four factors.

37. Simplify $\dfrac{x}{x-1}-\dfrac{x}{x+1}\div\left\{\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{1}{1-x^2}\right\}$

38. G.C.M. of (i.) $x^{2a}+y^{2b}-z^{2c}+2x^ay^b$ and $x^{2a}-y^{2b}+z^{2c}+2x^az^c$;
and (ii.) of $11x^4-9ax^3-a^2x^2-a^4$ and $13x^4-10ax^3-2a^2x^2-a^4$.

39. If x^3+qx+r and $3x^2+q$ have a common measure, then $\dfrac{r^2}{4}+\dfrac{q^3}{27}=0$.

40. H.C.D. of $a^3+a^2b-ab^2-b^3$, $a^3-2a^2b-ab^2+2b^3$, and $a^3-3ab^2+2b^3$.

41. H.C.D. and L.C.M. of $6x^2-5ax-6a^2$ and $4x^3-2ax^2-9a^3$.

42. Simplify (a.) $\dfrac{x+3y}{2(x-3y)}+\dfrac{x^2+9y^2}{x^2-9y^2}-\dfrac{x-3y}{2(x+3y)}$;

(β.) $\dfrac{x^4+a^2x^2-b^2x^2-a^2b^2}{x^4+a^2x^2+a^4}\div\dfrac{x^4-a^4}{x^3+a^3}$.

43. Find the H.C.F. of $x^3-15ax^2+48a^2x+64a^3$ and $x^2-10ax+16a^2$; and of $21x^2+38x+5$ and $129x^2+221x+10$.

44. Simplify $\dfrac{1}{x^2-3x+2}-\dfrac{2x}{x^2-4x+3}+\dfrac{1}{x^2-5x+6}$.

45. Reduce to their lowest terms the fractions

(i.) $\dfrac{(2x^2+5x+2)(x^3-3x^2-x+3)}{(x^3+6x^2+11x+6)(2x^2-x-1)}$; and (ii.) $\dfrac{8x^3-y^3+27z^3+18xyz}{2x-y+3z}$.

46. Simplify the fractions:

(i.) $\dfrac{2a^2+ab-b^2}{a^3+a^2b-a-b}$.

(ii.) $\dfrac{x}{(x-1)^2}-\dfrac{1}{(x+1)^2}-\dfrac{x(x^2+3)}{(1-x^2)^2}$.

(iii.) $\left\{\dfrac{a+bx}{a-bx}+\dfrac{b+ax}{b-ax}\right\}\div\left\{\dfrac{a+bx}{a-bx}-\dfrac{b+ax}{b-ax}\right\}$.

47. Find the Least Common Multiple of—

$$x^3 - y^3,\ x^3 + y^3 \text{ and } x^4 + x^2y^2 + y^4.$$

48. Find the Greatest Common Measure and Least Common Multiple of—

(i.) $(2a^2 + 3y^2)x + (2x^2 + 3a^2)y$, and $(2x^2 - 3a^2)y + (2a^2 - 3y^2)x$; also

(ii.) of $x^3 - 7x + 6$, $x^3 - 2x^2 - 5x + 6$, and $x^3 + 4x^2 + x - 6$.

49. Simplify—

(i.) $\dfrac{1}{6m - 2n} + \dfrac{1}{3m + 2n} - \dfrac{3}{6m + 2n}$; and (ii.) $\dfrac{x^3 - 41x - 30}{x^3 - 11x^2 + 25x + 25}$.

50. Divide—

$$\frac{\left(\dfrac{3x + x^3}{1 + 3x^2}\right)^2 - 1}{\dfrac{3x^2 - 1}{x^3 - 3x} + 1} \text{ by } \frac{\dfrac{9}{x^2} - \dfrac{33 - x^2}{3x^2 + 1}}{\dfrac{3}{x^2} - \dfrac{2(x^2 + 3)}{(x^3 - x)^2}}.$$

51. Reduce to their simplest forms—

(i.) $\dfrac{3x^2 + 2x + 4}{x^3 - 1} - \dfrac{x + 1}{x^2 + x + 1} - \dfrac{2}{x - 1}$; and (ii.) $\dfrac{x + y}{y} - \dfrac{2x}{x + y} + \dfrac{x^3 - x^2y}{y^3 - x^2y}$.

52. Find the condition that $x^3 + ax^2 + bx + c$ may be divisible by $x^2 + 2x + 3$ for all values of x; and that $x^3 + 5x^2 - ax + b$ may be divisible by both $x - 2$ and $x - 5$.

53. Reduce to their lowest terms— .

(i.) $\dfrac{x^8 + x^6y^2 + x^2y + y^3}{x^4 - y^4}$; and (ii.) $\dfrac{x^3 + (a + b)x^2 + (ab + 1)x + b}{bx^3 + (ab + 1)x^2 + (a + b)x + 1}$.

54. Prove that $\dfrac{1}{(y - z)^2} + \dfrac{1}{(z - x)^2} + \dfrac{1}{(x - y)^2} = \left\{\dfrac{1}{y - z} + \dfrac{1}{z - x} + \dfrac{1}{x - y}\right\}^2$.

55. Simplify $\left(\dfrac{4x^2}{y^2} - 1\right)\left(\dfrac{2x}{2x - y} - 1\right) + \left(\dfrac{8x^3}{y^3} - 1\right)\left(\dfrac{4x^2 + 2xy}{4x^2 + 2xy + y^2} - 1\right)$.

56. Find the condition that $x^3 + 7x^2 + ax + b$ may be divisible by both $x + 3$ and $x - 2$ for all values of x.

57. Find the G.C.M. of $3x^3 - 22x - 15$, $5x^4 - 17x^3 + 18x$,

and $x^6 - 3x^5 - x + 3$.

58. Find the H.C.D. of $19x^3 - 33x^2 - 9x - 2$ and $21x^3 - 39x^2 - 3x - 6$.

59. Find the condition that $x^3 + 4x^2 + 4x + b$ and $x^2 + 5x + c$ may have a common factor of the form $x + a$.

60. Find the L.C.M. of $x^3 - 3x^2 + 3x - 1$, $x^3 - x^2 - x + 1$, $x^4 - 2x^3 + 2x - 1$, and $x^4 - 2x^3 + 2x^2 - 2x + 1$.

III.—EVOLUTION.

1. To find a Square Root use the following standard example:—

$$
\begin{array}{rl}
 & a^2 + 2ab + b^2 (a + b \\
 & a^2 \\
2a + b) & \overline{\quad 2ab + b^2} \\
 & \quad \underline{2ab + b^2}
\end{array}
$$

Ex. (i.) Find $\sqrt[2]{(x^4 + 8x^3 - 64x + 64)}$.

$$
\begin{array}{rl}
 & x^4 + 8x^3 - 64x + 64 (x^2 + 4x - 8 \\
 & x^4 \\
2x^2 + 4x) & \overline{8x^3} \\
 & \underline{8x^3 + 16x^2} \\
2x^2 + 8x - 8) & - 16x^2 - 64x + 64 \\
 & \underline{- 16x^2 - 64x + 64}
\end{array}
$$

Ex. (ii.) Find $\sqrt{421521961}$.

$$
\begin{array}{rl}
 & \dot{4}2\dot{1}5\dot{2}1\dot{9}6\dot{1} (20531 \\
 & 4 \\
405) & \overline{2152} \\
 & 2025 \\
4103) & \overline{12719} \\
 & \underline{12309} \\
41061) & \overline{41061} \\
 & \underline{41061}
\end{array}
$$

2. The reason of the above method is seen by consideration of the inverse operation of finding the square of a given polynominal.

It has been already proved (I. § 2.) that $(a+b+c+d+\ldots)^2=a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d+$ etc. ... Thus, if we stop after any subtraction in the operation of finding the square root, we have taken away the square of that part of the root then obtained. For $a^2+(2a+b)b=(a+b)^2$, $a^2+(2a+b)b+(2a+2b+c)c=(a+b+c)^2$, etc. Hence when there is no remainder left, we have found an expression whose square is equal to the given expression.

3. When $\frac{x}{a}$ is a small fraction $\sqrt{a^2+x}=a+\frac{x}{2a}$ very nearly.

For $\left(a+\frac{x}{2a}\right)^2=a^2+2a\left(\frac{x}{2a}\right)+\left(\frac{x}{2a}\right)^2=a^2+x+\left(\frac{x}{2a}\right)^2.$

But if $\frac{x}{2a}$ is small fraction, $\left(\frac{x}{2a}\right)^2$ is still smaller;

$\therefore\ a^2+x+\left(\frac{x}{2a}\right)^2=a^2+x$ very nearly.

$\therefore\ \sqrt{a^2+x}=a+\frac{x}{2a}$ very nearly.

e.g. $\sqrt{26}=\sqrt{5^2+1}=5+\frac{1}{2\times 5}=5{\cdot}1$ nearly.

If $\sqrt{26}$ be got by the ordinary method, its value is 5·099 . . .

4. When $n+1$ figures of $\sqrt{N}$ have been obtained by the ordinary method, n more may be obtained by division only, if $2n+1$ be the whole number.

For if $\sqrt{N}=a+x$ where a, x contain $2n+1$, n digits respectively, then $\frac{N-a^2}{2a}=x+\frac{x^2}{2a}$, and $\frac{x^2}{2a}$ is a proper fraction, for x is $<10^n$ and $2a$ is $>10^{2n}$;

$\therefore\ x$, the part required $=$ the integer contained in the improper fraction $\frac{N-a^2}{2a}$.

e.g. $\sqrt{421521961}=20500+x$;

$$\therefore\ \frac{N-a^2}{2a}=\frac{1271961}{41000}=31+\frac{961}{41000}, \text{ see §§ 1, 2.}$$

$$\therefore\ x=31, \text{ and } \therefore\ \sqrt{421521961}=20531.$$

5. To find a Cube Root use the following example:—

$$
\begin{array}{lr|l}
3a+b & 3a^2 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & +3ab+b^2 & a^3 \\
\cline{2-2}
 & 3a^2+3ab+b^2 & \quad 3a^2b+3ab^2+b^3 \\
 & & \quad \underline{3a^2b+3ab^2+b^3}
\end{array}
$$

Ex. (i.) Find

$$\sqrt[3]{(x^6-6ax^5+21a^2x^4-44a^3x^3+63a^4x^2-54a^5x+27a^6)}.$$

$$
\begin{array}{ll|}
3x^2-2ax & 3x^4 \\
 & \qquad -6ax^3+4a^2x^2 \\
\cline{2-2}
 & 3x^4-6ax^3+4a^2x^2 \\
 & \qquad\qquad\quad 4a^2x^2 \\
\cline{2-2}
3x^2-6ax+3a^2 & 3x^4-12ax^3+12a^2x^2 \\
 & \qquad\qquad + \; 9a^2x^2-18a^3x+9a^4 \\
\cline{2-2}
 & 3x^4-12ax^3+21a^2x^2-18a^3x+9a^4
\end{array}
$$

$$
\begin{array}{|ll}
x^6-6ax^5+21a^2x^4-44a^3x^3+63a^4x^2-54a^5x+27a^6 & (x^2-2ax+3a^2 \\
x^6 & \\
\quad -6ax^5+21a^2x^4-44a^3x^3 & \\
\quad -6ax^5+12a^2x^4-\;\,8a^3x^3 & \\
\qquad\qquad 9a^2x^4-36a^3x^3+63a^4x^2-54a^5x+27a^6 & \\
\qquad\qquad \underline{9a^2x^4-36a^3x^3+63a^4x^2-54a^5x+27a^6} &
\end{array}
$$

Ex. (ii.) Find $\sqrt[3]{140851\cdot500427}$.

$$
\begin{array}{rr|l}
152 & 7500 & 14\dot{0}85\dot{1}\cdot50\dot{0}42\dot{7}(52\cdot03 \\
 & 304 & 125 \\
\cline{2-3}
 & 7804 & \;\;15851 \\
 & 4 & \;\;15608 \\
\cline{2-3}
15603 & 81120000 & \qquad 243500427 \\
 & 46809 & \\
\cline{2-2}
 & 81166809 & \qquad \underline{243500427}
\end{array}
$$

6. The following *numerical* results are important; for proof, see pp. 39-41. Imaginary results are rejected.

$\sqrt[4]{N}=\sqrt[2]{\sqrt[2]{N}}.$ *e.g.* $\sqrt[4]{625}=\sqrt[2]{\pm 25}=\pm 5.$

$\sqrt[6]{N}=\sqrt[2]{\sqrt[3]{N}}$ or $\sqrt[3]{\sqrt[2]{N}}.$ *e.g.* $\sqrt[6]{64}=\sqrt[2]{4}$ or $\sqrt[3]{\pm 8}=\pm 2.$

$\sqrt[8]{N}=\sqrt[2]{\sqrt[4]{N}}$ or $\sqrt[4]{\sqrt[2]{N}}.$ *e.g.* $\sqrt[8]{6561}=\sqrt[2]{\pm 9}$ or $\sqrt[4]{\pm 81}=\pm 3.$

EXAMPLES. III.

1. Find the square root of $x^4 - \frac{x}{2} + \frac{3x^2}{2} + \frac{1}{16} - 2x^3$.

2. Find the cube root of $1 - 6a + 21a^2 - 44a^3 + 63a^4 - 54a^5 + 27a^6$.

3. Find the fraction in its lowest terms which is the square root of
$$\frac{1 + 4x - 2x^2 - 12x^3 + 9x^4}{1 - 4x + 6x^2 - 4x^3 + x^4}.$$

4. Find the square root of $x^2 + 8x - \frac{64}{x} + \frac{64}{x^2}$.

5. Find $\sqrt[3]{379{\cdot}503424}$.

6. Find the cube roots of the following expressions :—

(i.) $\frac{x^3}{y^6} - 3\frac{x}{y^2} - \frac{y^6}{x^3} + 3\frac{y^2}{x}$.

(ii.) $\frac{a^3}{8} + \frac{8}{27a^6} + \frac{2}{3a^3} + \frac{1}{2}$.

(iii.) $x^3 - 12x^2 + 54x - 112 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}$.

7. Find the fourth root of
$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

8. Find the square roots of the following numbers :—
·00616225 ; 30470400 ; 77·0884 ; 6320·25 ; 1·0404612009.

9. Find the cube roots of the following numbers :—
994011·992 ; ·337153536 ; 53157376 ; 78402752 ;
29503629 ; 525557943 ; 676836152 ; 502459875.

10. Find the square root of $x^2 + \frac{1}{16x^2} + x - \frac{1}{4x} - \frac{1}{4}$.

11. Find the fourth root of

(i.) $4a^2b^2 + (a^2 + b^2)^2 + 4ab(a^2 + b^2)$,

and the cube root of

(ii.) $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

12. Find approximate values of the square roots of 145, 167, 226, and $30\frac{1}{2}$ by the method of § 3, p. 16.

13. Find the square roots of

(i.) $x^4+4ax^3+2a^2x^2-4a^3x+a^4$; (ii.) $x^6-22x^4+34x^3+121x^2-374x+289$;

and (iii.) $x^2+6xy+4xz+12yz+9y^2+4z^2$.

14. Simplify $\sqrt{\left\{x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1\right\}}$.

15. Find the square roots of the following numbers by the method of § 4, p. 16 :—

104142025 ; ·302133924.

16. Extract the square root of—

$$\frac{x^4}{16}+\frac{3x^3}{16}+\frac{25x^2}{64}+\frac{3x}{8}+\frac{1}{4}.$$

17. Simplify $\sqrt{4y^2+24xy^2+36x^2y^2} \div \sqrt[3]{1+9x+27x^2+27x^3}$.

18. Find the square root of

$$(x^2-3x+2)(x^2-4x+3)(x^2-5x+6).$$

19. Simplify $\sqrt{a^2-b^2}\ \sqrt{a^3+b^3}\ \sqrt{a^3-b^3}\ \sqrt{a^4+a^2b^2+b^4}$;

and $\sqrt{(x^2+6xy+4xz+12yz+9y^2+4z^2)}$.

20. Extract the square root of

$$25\frac{3}{7}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}.$$

IV.—EQUATIONS.

1. A *Root* of an equation which contains one unknown quantity is any quantity which when substituted for the unknown quantity in the equation makes the two sides equal to one another.

e.g. 2 is a root of $x^3-5x^2+7x=3x-4$, for $8-20+14=6-4$.

2. If a term be taken from one side of an equation to the other its sign must be changed.

For if equals be added to or taken from equals, the wholes or remainders are equal. Thus if $A \pm B = C$, then $A \pm B \mp B = C \mp B$, *i.e.* $A = C \mp B$.

Similarly, like terms may be struck out from the two sides of an equation, and the two sides may be multiplied or divided by the same quantity or raised to the same power.

3. In simultaneous simple equations if the roots are *determinate* there must be as many *independent* equations as there are unknown quantities.

It is often convenient to solve for $\frac{1}{x}, \frac{1}{y}$ . . . instead of x, y . . .

e.g. Solve $\frac{5}{x}+\frac{6}{y}=1$ and $5x+6y=xy$.

$$\left.\begin{aligned}\frac{30}{x}+\frac{36}{y}&=6\\ \frac{30}{x}+\frac{25}{y}&=5\end{aligned}\right\} \qquad \begin{aligned}&\therefore \frac{11}{y}=1\\ &\text{and} \therefore y=11 \text{ and } x=11.\end{aligned}$$

4. A quadratic equation $ax^2+bx+c=0$ cannot have more than *two* roots.

For *if possible* let the equation have three *unequal* roots α, β, γ, then by § 1,

$$\left.\begin{aligned}a\alpha^2+b\alpha+c&=0\\ a\beta^2+b\beta+c&=0\\ a\gamma^2+b\gamma+c&=0\end{aligned}\right\}$$

$\therefore a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$; and $\because$ α and β are by supposition unequal, $\alpha-\beta$ is not equal to zero: we may therefore divide out by $\alpha-\beta$.

$$\left.\begin{aligned}\text{Thus} \quad a(\alpha+\beta)+b&=0\\ \text{Similarly } a(\beta+\gamma)+b&=0\end{aligned}\right\}$$

$$\therefore a(\alpha+\gamma)=0 \text{ by subtraction;}$$

but a does not $=0$, $\therefore \alpha-\gamma=0$, $\therefore \alpha=\gamma$, which is contrary to hypothesis.

To find the two roots α, β, we proceed as follows :—

Take the constant term across and divide down by the co-efficient of x^2.

Thus $x^2+\frac{b}{a}x=-\frac{c}{a}$. Complete the square of left-hand side :

$$x^2+\frac{b}{a}x+\left(\frac{\text{co-efficient of } x}{2}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$i.e.\ x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Take the square root of both sides (p. 1, § 1):

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}, \therefore \alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

5. Therefore, in the equation $ax^2+bx+c=0$

(i.) if $b^2>4ac$, then α, β are *real* and *different*,
$b^2=4ac$, „ „ *real* and *equal*,
$b^2<4ac$, „ „ *imaginary* and *different.*

(ii.) if $b=0$, α and β are equal, but of opposite signs and are imaginary, unless a and c have opposite signs.

(iii.) If $a=0$, β is infinite and α is of the indeterminate form $\frac{0}{0}$, but by referring to the equation $0x^2+bx+c=0$ we find that $\alpha=-\frac{c}{b}$.

(iv.) If $c=0$, $\alpha=0$ and $\beta=-\frac{b}{a}$.

Example. Find the condition that $\frac{a}{x+b}+\frac{b}{x+c}+\frac{c}{x+d}=0$ may have only one finite root, and obtain that root.

Multiply up:

thus $a(x^2+cx+dx+cd)+b(x^2+bx+dx+bd)+c(x^2+bx+cx+bc)=0$

$\therefore (a+b+c)x^2+(ac+ad+b^2+bd+bc+c^2)x+acd+b^2d+bc^2=0.$

The condition therefore is $a+b+c=0$ by (iii.); and then the other root is clearly $x=-\frac{acd+b^2d+bc^2}{c(a+b+c)+ad+b^2+bd}=-\frac{acd+b^2d+bc^2}{ad+b^2+bd}$.

6. From § 4, $\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=-\frac{b}{a}$;

$$\text{and } \alpha\beta=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}.$$

Similarly, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$;

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c};$$

and $\alpha^3+\beta^3=(\alpha+\beta)\{(\alpha+\beta)^2-3\alpha\beta\}=\frac{3abc-b^3}{a^3}$: etc.

7. From § 6 we have $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$=a(x^2-\overline{\alpha+\beta}\,x+\alpha\beta)=a(x-\alpha)\,(x-\beta).$$

Thus if we can split a quadratic equation into two factors, each factor equated to zero will give a root.

$$e.g.\ x^2-12x+35=0,\ \therefore\ (x-5)(x-7)=0,$$

$$\therefore\ x-5=0, \text{ and } x-7=0,\ \therefore\ x=5 \text{ or } 7.$$

Conversely, to make a quadratic whose roots shall be p, q.

We get $(x-p)(x-q)=0$, *i.e.* $x^2-(p+q)x+pq=0$.

To make the equation whose roots are $\frac{3}{2}$ and -5: we get

$$(x-\tfrac{3}{2})(x+5)=0,\ i.e.\ x^2+\tfrac{7}{2}x-\tfrac{15}{2}=0, \text{ or } 2x^2+7x-15=0.$$

8. To prove that, if α, β are the roots of $x^2-px+q=0$, then x^2-px+q is identical with $(x-\alpha)\,(x-\beta)$.

Solving, we get

$$x^2-px+\left(\frac{p}{2}\right)^2=\frac{p^2-4q}{4};\ \therefore\ \alpha=\frac{p+\sqrt{p^2-4q}}{2},\ \beta=\frac{p-\sqrt{p^2-4q}}{2};$$

$$\therefore\ (x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta=x^2-px+q.$$

9. *Examples*:—

(i.) Find the condition that one root of $ax^2+bx+c=0$ may be thrice the other.

We have $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$. [See § 6, p. 21.]

Now let $\beta=3\alpha$, then $4\alpha=-\frac{b}{a}$ and $3\alpha^2=\frac{c}{a}$,

$$\therefore\ 3\left(-\frac{b}{4a}\right)^2=\frac{c}{a};\ \therefore\ 3b^2-16ac=0.$$

(ii.) If the equation $x^2+2(1+k)x+k^2=0$ has equal roots, what must be the value of k?

$$x^2+2(1+k)x+(1+k)^2=1+2k+k^2-k^2,$$

$$\therefore\ x=-(1+k)\pm\sqrt{1+2k}.$$

If the roots are equal the square root must vanish.

$$i.e.\ 1+2k=0,\qquad \therefore\ k=-\tfrac{1}{2}.$$

(iii.) If the roots of $x^2+ax-p=0$ be b, c, and of $x^2+bx-q=0$ be c, a, and if $2(p+q+r)=a^2+b^2+c^2$, prove that the equation whose roots are a, b is $x^2+cx-r=0$.

By § 6 $b+c=-a$, $c+a=-b$,

$$\therefore c=-(a+b)\text{ ; and } bc=-p,\ ca=-q.$$

But $2r=a^2+b^2+c^2-2(p+q)=a^2+b^2+(a+b)^2+2c(a+b)$

$$=a^2+b^2+a^2+2ab+b^2-2(a+b)^2=-2ab,\ \therefore ab=-r$$

$\therefore$ the equation whose roots are a, b is $x^2+cx-r=0$.

(iv.) If the roots of $x^2+px+q=0$ are real, prove that those of $x^2+3px+2p^2+q=0$ are real also.

Condition that roots of second equation may be real is

$$(3p)^2-4(2p^2+q)>0.\quad \text{See § 5.}$$

i.e. that $p^2-4q>0$, which is the condition that roots of first equation may be real.

(v.) If the roots of $ax^2+2bx+c=0$ be α, β, and those of $Ax^2+2Bx+C=0$ be $\alpha+\gamma$, $\beta+\gamma$, prove that $\dfrac{b^2-ac}{a^2}=\dfrac{B^2-AC}{A^2}$.

$$\frac{B^2-AC}{A^2}=\left(\frac{B}{A}\right)^2-\frac{C}{A}=\left(-\frac{\alpha+\gamma+\beta+\gamma}{2}\right)^2-(\alpha+\gamma)(\beta+\gamma)=\left(\frac{\alpha+\beta}{2}\right)^2$$

$$+(\alpha+\beta)\gamma+\gamma^2-\alpha\beta-(\alpha+\beta)\gamma-\gamma^2=\left(\frac{\alpha+\beta}{2}\right)^2-\alpha\beta=\left(-\frac{b}{a}\right)^2-\frac{c}{a}=\frac{b^2-ac}{a^2}.$$

(vi.) If x_1, x_2 are the roots of $ax^2+bx+c=0$, find the value of $(b+ax_1)^{-2}+(b+ax_2)^{-2}$ in terms of a, b, c.

$$\text{By § 4, } (b+ax_1)^{-2}+(b+ax_2)^{-2}=\left(b+\frac{-b+\sqrt{b^2-4ac}}{2}\right)^{-2}$$

$$+\left(b+\frac{-b-\sqrt{b^2-4ac}}{2}\right)^{-2}=\frac{4}{(b+\sqrt{b^2-4ac})^2}+\frac{4}{(b-\sqrt{b^2-4ac})^2}$$

$$=\frac{4\{2b^2-4ac-2b\sqrt{b^2-4ac}+2b^2-4ac+2b\sqrt{b^2-4ac}\}}{16a^2c^2}=\frac{b^2-2ac}{a^2c^2}.$$

(vii.) If $4x^2+1=5x$, show that $4x^2-1=\pm 3x$, and hence solve the given equation.

Square: thus $16x^4+8x^2+1=25x^2, \therefore 16x^4-8x^2+1=9x^2, \therefore 4x^2-1=\pm 3x$

$$\therefore \text{ by subtraction } 2=(5\mp 3)x=2x \text{ or } 8x, \therefore x=1 \text{ or } \frac{1}{4}.$$

(viii.) Form an equation whose roots are the Arithmetic and Harmonic Means between those of $x^2-px+q=0$.

$$\text{A.M.}=\frac{a+\beta}{2}=\frac{p}{2}, \text{ and H.M.}=\frac{2a\beta}{a+\beta}=\frac{2q}{p}.$$ See § 7, p. 53.

$$\therefore \text{ the equation is } 2px^2-(p^2+4q)x+2pq=0.$$

(ix.) Form the quadratic equation whose roots are $-3+\sqrt{2}$ and $-3-\sqrt{2}$.

$$(x+3-\sqrt{2})(x+3+\sqrt{2})=(x+3)^2-2=x^2+6x+7$$

$$\therefore \text{ the required equation is } x^2+6x+7=0.$$

EXAMPLES. IV. (A.)

1. For what values of a will $x^2-ax+a+3=0$ have *equal* roots?

2. If a, β are the roots of $x^2+px+q=0$, and if $\beta=\frac{1}{a^2}$, prove that $q^3+pq+1=0$.

3. If a, β are the roots of $x^2-px+q=0$, and a^4, β^4 the roots of $x^2-Px+Q=0$; express P, Q in terms of p, q.

4. If a, β are the roots of $x^2+px+1=0$, and γ, δ the roots of $x^2+qx+1=0$, prove that $(a-\gamma)(\beta-\gamma)(a+\delta)(\beta+\delta)=q^2-p^2$.

5. Form the quadratic equations whose roots are (i.) $-5\pm 6\sqrt{-1}$, and (ii.) $\frac{-1\pm\sqrt{-3}}{2}$.

6. Show that no real values of x satisfy the equation $x^2+a^2=6a-8x$, unless a lie between -2 and $+8$.

7. If the equation $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{m}{(x+a)(x+b)(x+c)}$ have *equal* roots, find m in terms of a, b, c.

8. If a, β be the roots of $3x^2-2x+1=0$, show that $\frac{a^3+\beta^3}{a^{-2}+\beta^{-2}}=\frac{5}{27}$.

9. Solve the equation $x-\sqrt{5x+10}=8$, showing that it leads to a quadratic for the determination of x. Explain the reason why both of these values of x will not satisfy the given equation, although they satisfy the quadratic from which they are found.

10. Find the condition that the roots of $a^2x^2+b^2x+c^2=0$ may be the squares of the roots of $ax^2+bx+c=0$.

EXAMPLES. IV. (B.)

Solve the following equations :—

1. $\sqrt{x+1}+\sqrt{x+6}=5$.

2. $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$.

3. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$.

4. $x^2=21+\sqrt{x^2-9}$.

5. $x^4-13x^2+36=0$.

6. $x^{10}-31x^5-32=0$.

7. $x+\dfrac{1}{x}+\dfrac{1}{x+\dfrac{1}{x}}=2\frac{9}{10}$.

8. $\sqrt{\dfrac{x-4}{x+8}}=\dfrac{8}{125}\cdot\dfrac{x+8}{x-4}$.

9. $x^{-3}+\dfrac{1}{x\sqrt{x}}=2$.

10. $(7+4\sqrt{3})x^2+(2+\sqrt{3})x=2$.

11. $\dfrac{1}{x+a+b}=\dfrac{1}{x}+\dfrac{1}{a}+\dfrac{1}{b}$.

12. $(x+1)^2=x+3\sqrt{3x^2+3x-11}$.

13. $(x+1)(x+2)(x+3)=2(7x^2+2)+(x-1)(x-2)$.

14. $\left(x-\dfrac{1}{x}\right)\left(x-\dfrac{4}{x}\right)\left(x-\dfrac{9}{x}\right)=(x-1)(x-2)(x-3)$.

15. $2x+5y=22, \quad 4x^2-25y^2=44$.

16. $x+y=12, \quad x^3+y^3=468$.

17. $xy=1225,\quad \sqrt{x}+\sqrt{y}=12.$

18. $x^2-xy=2,\quad 2x^2+y^2=9.$

19. $x^2+y^2+x+y=18,\quad xy=6.$

20. $x+y=\frac{15}{4},\quad x-y=xy.$

21. $yz=a(y+z),\quad zx=b(z+x),\quad xy=c(x+y).$

22. $x^3y=2z,\quad y^3z=9x,\quad xyz=6.$

23. $(z+x)(x+y)=9,\quad (x+y)(y+z)=16,\quad (y+z)(z+x)=25.$

24. $x^2+y^2+z^2=29,\quad yz+zx+xy=26,\quad x+y=5.$

25. $12\times 2^{x+1}-4^x=128.$

26. $(x+y)(x^{\frac{1}{2}}-y^{\frac{1}{2}})=13,\quad (x-y)(x^{\frac{1}{2}}+y^{\frac{1}{2}})=25.$

27. $ax+by=a^2+b^2,\quad a^2x+b^2y=a^3+b^3.$

28. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{13}{6}.$

29. $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}.$

30. $x(x+y+z)=45,\quad y(x+y+z)=70,\quad z(x+y+z)=105.$

31. $\dfrac{(x-1)(x-2)(x-6)}{(x-3)^3}=1.$

32. $\dfrac{2x+5}{2x-3}-\dfrac{3x-1}{x+1}=\dfrac{5}{3}.$

33. $$\left.\begin{array}{r}(x+1)^2+(x+1)(y+2)+(y+2)^2=133,\\ (x+1)+\sqrt{(x+1)(y+2)}+(y+2)=19.\end{array}\right\}$$

34. $(2x+1{\cdot}5)(3x-2{\cdot}25)=(2x-1{\cdot}125)(3x+1{\cdot}25).$

35. $\dfrac{x}{2(x+3)}-2\tfrac{5}{24}=\dfrac{x^2}{x^2-9}-\dfrac{8x-1}{4(x-3)}.$

36. $x^2(3ax-2by)=ab^3,\quad xy(3by-2ax)=a^2b^2.$

37. $\sqrt{1+2x}+\sqrt{1-2x}=\dfrac{12}{5\sqrt{1+2x}}.$

38. $\dfrac{x^2}{2y}+\dfrac{4y^2}{x}=\dfrac{7}{2},\quad 1+xy=0.$

39. $\dfrac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}=\dfrac{7+3\sqrt{5}}{2}.$

40. $\frac{a}{b}x^2+\left(1+\frac{b}{a}\right)x-\frac{1}{2}=\frac{b}{a}x^2-\left(1-\frac{a}{b}\right)+x+\frac{1}{2}.$

41. $\frac{7x+5y-12}{7x-5y+12}=\frac{29}{6},\quad \frac{7x}{5}-\frac{5y}{7}=\frac{4x+8y-6}{35}.$

42. $\frac{x+5}{x-3}-\frac{x-3}{x+5}=\frac{80}{9}.$

43. $(x-3)^3+(x-4)^3=43\ \{(x-3)^2-(x-4)^2\}.$

44. $\frac{b-c}{x-a}+\frac{c-a}{x}+\frac{a-b}{x-c}=0.$

45. $(x+a)(a^2-ax+b^2)=a^3+bx^2.$

46. $\frac{x^3}{y}-\frac{9}{x}=\frac{8x}{\sqrt{y}},\quad \frac{y}{x^2}+\frac{1}{\sqrt{y}}=\frac{10x^2}{81y^2}.$

47. $\frac{1}{x-a}+\frac{2}{x+b}+\frac{4}{a+b}=0.$

48. $2(x-1)(x-2)-\sqrt{x^2-3x+6}=20.$

49. $y^2+xy=\sqrt{2}-1,\quad x^2+y^2=2.$

50. $x(y+z-x)=39-2x^2,\quad y(z+x-y)=52-2y^2,\quad z(x+y-z)=78-2z^2.$

51. $2(x^2+xy)=3y$, and $y^2+xy=6x$.

52. $\sqrt{x}+\sqrt{4+x}=\frac{4}{\sqrt{x}}.$

53. $\frac{x}{10}-\frac{1}{5}\cdot\frac{x+1}{x+2}+\frac{3-2x}{20}=0.$

54. $\frac{x+1}{x-1}-\frac{1}{2}\cdot\frac{x-1}{x+1}=1\frac{3}{4}.$

55. $x^2+y^2=8,\quad 2xy-y^2=4.$

56. $\frac{x-\frac{1}{5}}{\frac{1}{5}}-\frac{x-\frac{1}{3}}{\frac{1}{3}}=\frac{x-\frac{1}{4}}{\frac{1}{4}}.$

57. $\frac{10}{2x+1}-\frac{15}{3x+1}=\frac{1}{x+1}-\frac{6}{6x+1}.$

58. $\sqrt{5x-1}+\sqrt{7x-6}=\sqrt{20x+25}.$

59. $2x^2-3xy+y^2=6,\ x-y=1.$

60. $\frac{x^2}{y^2}+\frac{y^2}{x^2}=2\frac{81}{400},\quad x^2+y^2=41.$

61. $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$.

62. $\dfrac{\sqrt{x^2+y^2}+\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}-\sqrt{x^2-y^2}}=\dfrac{5+\sqrt{7}}{5-\sqrt{7}},\quad \dfrac{x^2}{16}+12=\dfrac{8y}{3}$.

63. $x+y+x^2+y^2=18,\quad xy=6$.

64. $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4$.

65. $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=c$.

66. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{b}{x}$.

67. $\sqrt{x}-\sqrt{\dfrac{a}{x}}=\sqrt{a+x}$.

68. $a+x+\sqrt{2ax+x^2}=b$.

69. $\dfrac{2\sqrt[3]{17x-26}}{3}+\dfrac{3}{4}=\dfrac{25}{12}$.

70. $\dfrac{\sqrt[4]{x+12}}{2}+\dfrac{3}{4}=1$.

71. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$.

72. $6x+a:4x+b::3x-b:2x-a$.

73. $16x^2+3p^2=16px$.

74. $\left.\begin{aligned}\dfrac{5x+6}{10}-\dfrac{11y-5}{21}&=11,\\ \dfrac{1}{25}(55y-12)&=\dfrac{7x}{5}-37.\end{aligned}\right\}$

75. $\dfrac{y}{x}-\dfrac{x}{y}=\dfrac{x+3}{y+3}=\dfrac{x+y}{xy}$.

76. $\left.\begin{aligned}(a-b)x+(a+b)y&=2(a^2-b^2),\\ ax-by&=a^2+b^2.\end{aligned}\right\}$

77. $(x-4)^3+(x-5)^3=31\{(x-4)^2-(x-5)^2\}$.

78. $\dfrac{3+10x}{5x-7} = 9 - \dfrac{7x+17}{x+1}$.

79. $\dfrac{(x+a)(x+b)}{x+a+b} = \dfrac{(x+c)(x+d)}{x+c+d}$.

80. $3x + 2x^{\frac{1}{2}} - 1 = 0$.

81. $x^2 + xy = 140, \quad y^2 + xy = 56$.

82. $x^2 + 3y^2 = 43, \quad x^2 + xy = 28$.

83. $xz = y^2, \quad x + y + z = 21, \quad x^2 + y^2 + z^2 = 189$.

84. $x^2y + y^2x = 30, \quad \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$.

85. $3x^{\frac{5}{2}} + x^{\frac{5}{4}} = 3104$.

86. $x^{\frac{3}{2}} + x^{\frac{3}{4}} = 756$.

87. $3x^{\frac{1}{2}} - 85x^{\frac{1}{4}} + 592 = 0$.

88. $2x\sqrt[3]{x} - 3x\sqrt[3]{\dfrac{1}{x}} = 20$.

89. $2x^4 - x^2 + 104 = 600$.

90. $x + y + z = 6, \quad x^2 + y^2 - z^2 + 2xy = 24, \quad x - y = 1$.

91. $x^2 + xy + y^2 = 7(x+y), \quad x^2 - xy + y^2 = 9(x-y)$.

92. $\sqrt{x^2 - 12x + 36} + \sqrt{x^2 - 7x + 6} = \sqrt{x^2 + 9x - 90}$.

93. $x^2 - 3x + 2\sqrt{x^2 - 3x + 1} = 2$.

94. $x^2 + 5xy + 6y^2 = 154, \quad 15x^2 - 8xy + y^2 = 264$.

95. $x + y + \dfrac{2xy}{x+y} = 13, \quad x^2 + y^2 + \dfrac{4x^2y^2}{(x+y)^2} = 61$.

96. $\dfrac{x}{a} - \dfrac{y}{b} = 2, \quad \sqrt{2a(b-y)} = \sqrt{b}(\sqrt{a} + \sqrt{x})$.

97. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2}, \quad x^2 + y^2 = 90$.

98. $xy + yz + zx = 11, \quad 2xy + 3xz + 5yz = 31, \quad 7xy + 5xz + 7yz = 71$.

99. $\dfrac{x+y}{x-y} + 10\left(\dfrac{x-y}{x+y}\right) = 7, \; xy^3 = 3$.

100. $\dfrac{x^2 + 8x + 15}{x^2 + 9x + 20} + \dfrac{x^2 + 4x + 3}{x^2 + 5x + 6} = \dfrac{19}{12}$.

EXAMPLES. IV. (C.)—Problems.

1. Find a number such that if three-eighths of it be subtracted from 20, and five-elevenths of the remainder from one-fourth of the original number, 12 times the second remainder shall be one-half the original number.

2. A farmer bought a certain number of sheep for £57. Having lost 8 of them, he sold the remainder at 8s. a head more than they cost him, in order to make up the deficiency. How many did he buy?

3. A space in the shape of a rectangle with semicircular ends is surrounded by 100 yards of palisading, and contains 546 square yards. Find the width and extreme length of the enclosure, assuming that the area of a circle $= \frac{22}{7} \times (\text{radius})^2$, and the circumference $= \frac{22}{7} \times (\text{diameter})$.

4. A merchant bought some pieces of silk for £221. Had he bought 4 pieces more for the same money, he would have paid £4 less for each piece. How many did he buy?

5. Divide £1, 6s. 2d. among 5 men, 11 women, and 8 boys, so that 3 men may have as much as 2 women and 2 boys together, and 5 women as much as 7 boys.

6. The commercial discount on a certain sum for one year is to the true discount as 31 : 30. Find the rate per cent.

7. A father gave his son a certain sum, telling him that at the end of every year he would give him as much as he then had left: the son spent £100 a year, and at the end of 4 years had nothing left. How much did he receive at first?

8. A grocer gains 20 per cent. by selling at 2s. a lb. a mixture formed by mixing 7 lbs. of common tea with 2 lbs. of a better kind; but if he had mixed 7 lbs. of the latter with 2 lbs. of the former kind, he would have lost 20 per cent. by selling the mixture at that price. What did each kind of tea cost him per lb.?

9. A does a piece of work in two-thirds of the time in which B can do it, but A receives 2s. a day more. The work costs 20s. more if B does it than if A does it, and, if both work together, it costs 96s.; find the daily wages of each.

10. A bill before Parliament was lost on a division, there being 600 votes recorded. Afterwards, there being the same voters, it was carried by twice as many votes as it was before lost by, and the new majority was to the former as 5 : 4. How many members changed their minds?

11. The difference of two numbers is 6, and the difference of their squares exceeds 32 times the sum of their reciprocals by 100 times their sum divided by 3 times their difference. Find them.

12. The hypotenuse of a right-angled triangle is less than the sum of the other two sides by 8 ft., and the area = 180 sq. ft. Find all the sides.

13. The sum of £11, 7s. 6d. is made up of a certain number of sovereigns, twice as many half-crowns, five times as many shillings, and ten times as many threepenny pieces. Find the numbers of each coin.

14. In a journey of 180 miles, an increase of 5 miles per hour in the rate of the train would diminish the time required by 30 minutes. What is the rate of the train?

15. The numerator of one fraction is the same as the denominator of another; their product is 12, the sum of their numerators 41, and the sum of their denominators 8; find the fractions.

16. A train A starts to go from P to Q, two stations 240 miles apart, and travels uniformly. An hour later another train B starts from P, and, after travelling for two hours, comes to a point that A had passed 45 minutes previously. The pace of B is now increased by 5 miles an hour; and it overtakes A just on entering Q. Find the rates at which they started.

17. A carrier charges 3d. each for all parcels not exceeding a certain weight; and on heavier parcels he makes an additional charge for every 7 lbs. above that weight. The charge for half a cwt. is 1s. 3d., and the charge for 9 stones is five times that for 1 qr. What is the scale of charges?

18. A's annual income is a half of B's, and B spends £60 a year more than A does. At the end of two years A has saved £200 and B £600. What are their yearly incomes?

19. An officer, in forming his regiment into a solid square, found he had 76 too many, and, upon increasing the side of the square by 2, he had 88 too few; of how many did the regiment consist?

20. On a certain road the number of telegraph posts per mile is such that if there were one less in each mile the interval between the posts would be increased by $2\frac{14}{15}$ yards. Find the number of the posts per mile.

21. Two casks, each containing 20 gallons, are filled, one with water, the other with spirit, (x) gallons are drawn from each cask, mixed, and the casks are again filled up with the mixture; when this is done a second time it is found that the quantity of spirit in the latter cask is to the quantity in the former as 5 to 3. Find (x).

22. Find 3 numbers such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the products shall be 26, 50, and 56.

23. A ship did a measured mile in $4'$ less on a perfectly calm day under steam than on a windy day under sail; also on the windy day, using steam in addition to sail, she did the mile in $2\frac{2}{3}'$. What was the ship's time over the mile under steam on the calm day?

24. How soon after three o'clock are the hour and minute hands of a clock again at right angles to each other?

25. A person started at a certain pace to walk to a railway station five miles off, allowing himself just time to catch a train: after walking one mile, he stopped for twenty minutes, but by walking one mile an hour faster, he was still in time for the train. At what pace did he start?

26. A sets out from London to York, and B at the same time from York to London: they travel uniformly, and A reaches York 16 hours and B London 36 hours after they have met on the road. What time did each take over his journey?

27. A broker bought as many railway shares as cost him £1875, he reserved 15 of the shares and sold the remainder for £1740, gaining £4 a share on their cost price. How many shares did he buy, and what price did he pay for each?

28. The same amount of tax was paid on the profits of a business when the income tax was 6d. per £1, as when the profits were £300 more, and the tax 2d. less per £1. Find the profits in each case.

29. A person owes certain sums of money to two creditors, and he finds that if he pays them respectively sums of money in the ratio of 9 : 17 the sums unpaid will be 10 per cent. and 15 per cent. of the two debts, and will together amount to £52. Find the amount of each debt.

30. A wheel revolving with a given velocity drives a smaller wheel by means of cogs. If there had been 4 more cogs in each wheel, the velocity of the smaller wheel would have been one-sixth less, and if there had been 4 less in each, the velocity would have been one-third greater. Find the number of cogs in each wheel.

31. A bar of metal 9 inches wide, 2 inches thick, and 8 ft. long, weighs 1 lb. to the cubic inch. Find the length and thickness of another bar of the same metal, width, and solid content, if 2 inches cut off from the end weighs 27 lbs.

32. Find the sums of money with which each of 4 persons A, B, C, D began to play, if after A has won half of B's, B a third of C's, and C a fourth of D's, each has £12.

33. Find two numbers such that their difference = 4, and that twice their product = cube of the less.

34. Divide £100 between 3 men, 5 women, 4 boys, and 3 girls, so that each man has as much as a woman and a girl, each woman as much as a boy and a girl, and each boy half as much as a man and a girl together.

35. Some bees were sitting on a tree: at one time the square root of half their number flew away, and afterwards $\frac{8}{9}$ of the whole number. Then only 2 bees were left. How many were there in all?

36. A farmer bought 5 oxen and 12 sheep for £63, and for £90 could have bought 4 more oxen than he could have bought sheep for £9. What did he pay for each?

37. A runs 200 yards in $25\frac{1}{2}$ seconds, B runs the same distance in 26 seconds: how many yards must A give B in a 200 yards race in order that B may win by one yard?

38. The difference between the sum and difference of two numbers is half the sum of their sum and difference. What is the ratio of the larger to the smaller?

39. A vessel is filled with a mixture of 1 gallon of brandy and 9 gallons of water. If a gallon of the mixture be drawn off every day and the vessel filled up with brandy, how much brandy will there be in the vessel at the end of 5 days?

40. At what time between 8 and 9 are the hands of a clock at right angles?

41. Find the price of eggs when a reduction of 1d. a dozen gives an extra dozen for 11s.

42. Divide 128 into two parts so that the first $+7$, the second -7, the third $\times 7$, and the fourth $\div 7$ may be all equal.

43. Find two numbers such that the greater is to the less as their sum is to 90 and also as their difference is to 18.

44. A sets out from L to Y, and at the same time B sets out from Y to L. They meet on the road and find that A has travelled 30 miles more than B. A expects to reach Y in 4 days, and B to reach L in 9 days, each travelling at the same rate as before. Find the distance from L to Y.

45. The tail of a fish weighed 3 lbs. Its head weighed as much as its tail and half its body, and its body weighed as much as its head and tail. What did the fish weigh?

46. Find when next after twelve o'clock the hour and minute hands of a clock are together.

47. The fore-wheel of a carriage makes 6 revolutions more than the hind in going 120 yards; but if the circumference of each wheel were increased by 3 feet, the fore-wheel would make only 4 revolutions more than the hind in the same space. What is the circumference of each wheel?

V.—RATIO, PROPORTION, AND VARIATION.

1. Ratio is defined by Euclid as the mutual relation of two magnitudes of the same kind to one another in respect of quantity (see Bk. V. Def. iii.). In Algebra the ratio of $a : b$ is measured by the ratio-fraction $\frac{\text{antecedent}}{\text{consequent}} = \frac{a}{b}$; and the ratio is said to be one of *greater* or *less inequality*, or of *equality*, according as a is $>$ or $<$, or $= b$, *i.e.* as $\frac{a}{b}$ is $>$ or $<$, or $=1$.

To *compare* ratios we compare the corresponding ratio-fractions.

Ex. Show that the ratio $a^2 - x^2 : a^2 + x^2$ is $>$ the ratio $a - x : a + x$, where a, x are positive.

Compare $\frac{a^2-x^2}{a^2+x^2}$ and $\frac{a-x}{a+x}$, *i.e.* $\frac{a^2-x^2}{a^2+x^2}$ and $\frac{a^2-x^2}{a^2+2ax+x^2}$, and the result is evident.

To *compound* ratios we multiply together the ratio-fractions.

2. A ratio of greater inequality is diminished, and of less inequality is increased by adding the same positive quantity to both its terms.

For, let $\frac{a}{b}$ be the original ratio, and let some quantity d be added to both its terms, so that the new ratio is $\frac{a+d}{b+d}$.

Then $\frac{a+d}{b+d}$ is $>$ or $<\frac{a}{b}$ according as $\frac{ab+bd}{(b+d)b}$ is $>$ or $<\frac{ab+ad}{(b+d)b}$ (the fractions being brought to their L.C.D.), *i.e.* as bd is $>$ or $<ad$, *i.e.* as b is $>$ or $<a$, *i.e.* as given ratio $a : b$ is of less or greater inequality.

As d is indefinitely increased it will be seen that the ratio in either case approaches a ratio of equality.

3. Given any number of ratios equal to one another, to prove relations connecting their antecedents and consequents.

Assume $\frac{a}{b}=\frac{c}{d}=\ldots=k$, and substitute for all the *antecedents*, thus $a=bk$, $c=dk\ldots$, and get rid of k from the result.

Ex. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\left(\frac{a^3+ace+c^2e}{b^3+bdf+d^2f}\right)^{\frac{1}{3}}=$ each of these ratios.

$$\left(\frac{a^3+ace+c^2e}{b^3+bdf+d^2f}\right)^{\frac{1}{3}}=\left(\frac{b^3k^3+bkdkfk+d^2k^2fk}{b^3+bdf+d^2f}\right)^{\frac{1}{3}}=\left\{\frac{(b^3+bdf+d^2f)k^3}{b^3+bdf+d^2f}\right\}^{\frac{1}{3}}=k.$$

4. When are four quantities a, b, c, d said to be *proportionals?*

(i.) *Algebraical Definition.*—a, b, c, d are proportionals when the ratio of $a : b=$ the ratio of $c : d$, that is, when $\frac{a}{b}=\frac{c}{d}$.

(ii.) *Euclid's Definition.*—(Bk. V. Def. v.), a, b, c, d are proportionals if, when any equimultiples whatever be taken of a and c, and any whatever of b and d, the multiple of a is $>=$ or $<$ the multiple of b, according as the multiple of c is $>=$ or $<$ the multiple of d.

We may easily deduce (ii.) from (i.) thus: $\frac{a}{b}=\frac{c}{d}$, multiply both by $\frac{m}{n}$ where m and n are any whole numbers: thus $\frac{ma}{nb}=\frac{mc}{nd}$, and therefore ma is $>=$ or $<nb$ as mc is $>=$ or $<nd$.

To deduce (i.) from (ii.) we must use a *reductio ad absurdum* proof.

If $\frac{a}{b}$ does not equal $\frac{c}{d}$, let it be the greater of the two, and take some fraction $\frac{m}{n}$ lying between them where m and n are *integers*, as must be always possible if m, n be taken large enough.

Then $\frac{a}{b} > \frac{m}{n} > \frac{c}{d}$, $\therefore na > mb$ and $nc < md$, which is contrary to (ii.)

5. If $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, therefore $ad = bc$, *i.e.* the product of the *extremes* = the product of the *means*.

And $\because ad = bc$, $\therefore \frac{ad}{cd} = \frac{bc}{cd}$, *i.e.* $\frac{a}{c} = \frac{b}{d}$; $\therefore a : c :: b : d$, and similarly $b : a :: d : c$, etc.

And *conversely*, if $ad = bc$, then $a : b :: c : d$.

Again, by means of § 3, we readily get such relations as

$$a \pm b : b :: c \pm d : d, \text{ and } \frac{a+b}{a-b} = \frac{c+d}{c-d},$$

the latter of which is of great use in the solution of equations. (See *Euclid*, Bk. V. Defns. xiii. etc.)

6. Three quantities a, b, c are said to be in *continued proportion* if $a : b :: b : c$.

Then $b^2 = ac$, and $\therefore b = \sqrt{ac}$, and b is the *mean proportional* or *Geometric mean* between a and c. Also c is the *third proportional* to a and b, and a the *third proportional* to c and b.

Again $\frac{a}{b} = \frac{b}{c}$, $\therefore \frac{a}{b} \cdot \frac{a}{b} = \frac{b}{c} \cdot \frac{a}{b}$, $\therefore a^2 : b^2 :: a : c$, that is, $a : c$ in the *duplicate ratio* of a to b. (See *Euclid*, Bk. V. Def. x.)

Similarly, $a : b :: \sqrt{a} : \sqrt{c}$, *i.e.* in the *sub-duplicate* ratio of a to c.

7. If x varies as y *directly*, then $x : a :: y : b$ where a, b are some definite values of x, y. Thus $xb = ay$, and $\therefore x = \frac{a}{b}y$.

Therefore if $x \propto y$ *directly*, $x = ky$ where k is some *constant* quantity.

If $x \propto y$ *inversely*, then $x \propto \frac{1}{y}$, and $\therefore x = \frac{k}{y}$.

When $x \propto$ product of y and z, *i.e.* when $x \propto yz$, then x is said to vary as y and z *jointly*; and then $x=kyz$.

8. If $x \propto y$ when z is constant, and as z when y is constant, then $x \propto y$ and z *jointly* when both are variable.

For let $x=kyz$, where k is at present unknown and might contain either y or z, or both.

Then when y is constant and z varies, $x=ky\,.\,z$, and $\therefore ky$ does not contain z, *i.e.* k does not contain z. Similarly, k does not contain y, and is therefore *constant*, $\therefore x=(\text{constant}) \times yz$, $\therefore x \propto yz$.

Ex. The area of a parallelogram $= \sin\theta\,.\,xy$, where x, y are two adjacent sides and θ the $\angle$ between them: therefore the area of a parallelogram varies as either side when the other is fixed.

It will be noticed that this is the algebraical form of the Double Rule of Three.

EXAMPLES. V.

1. If $\dfrac{a+b}{a-b}=\dfrac{c}{d}$, prove that $\dfrac{c+d}{c-d}=\dfrac{a}{b}$.

2. If $a:b=c:d$, show that $a^2+b^2:ab=c^2+d^2:cd$.

3. The area of a certain rectangle is half that of the circumscribing circle. Prove that its sides are in the ratio of $5+\sqrt{3}:5-\sqrt{3}$, assuming that the area of a circle $=\frac{22}{7}\times(\text{radius})^2$.

4. Two casks A and B contain mixtures of wine and water, A in the ratio of $8:3$ and B in the ratio of $5:1$. In what ratio must liquid be drawn from each cask to give a mixture in the ratio of $4:1$?

5. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, then each of these fractions is equal to

$$\left(\frac{lx^2+my^2+nz^2}{la^2+mb^2+nc^2}\right)^{\frac{1}{2}}$$

6. Prove that

$$9a^2-4b^2:15a^2-31ab+14b^2::15a^2+31ab+14b^2:25a^2-49b^2.$$

7. The pressure of the wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on a square foot is 1 lb. when the wind is moving at a rate of 15 miles per hour. Find the velocity of the wind when the pressure on a square yard is 16 lbs.

8. Prove that $a^3+b^3 : a+b = a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5 : a^3-b^3$.

9. If $\dfrac{bx-ay}{cy-az}=\dfrac{cx-az}{by-ax}=\dfrac{z+y}{x+z}$, then will each of these fractions $=\dfrac{x}{y}$ unless $b+c=0$.

10. If $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$, prove that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

11. The duration of a railway journey varies directly as the distance and inversely as the velocity. The velocity varies directly as the square root of the quantity of coal used per mile and inversely as the number of carriages in the train. In a journey of 25 miles in half an hour with 18 carriages 10 cwt. of coal is required. How much coal will be consumed in a journey of 15 miles in 20 minutes with 20 carriages?

12. If $a:b::c:d$, prove that $a(a+b+c+d)=(a+b)(a+c)$, and that $a^3b^2 : c^3d^2 :: a^5+b^5 : c^5+d^5$.

13. If $\dfrac{y+z}{ay+bz}=\dfrac{z+x}{az+bx}=\dfrac{x+y}{ax+by}$, prove that each of these fractions $=\dfrac{2}{a+b}$ unless $x+y+z=0$.

14. If the first of 6 quantities be to the second as the third to the fourth, and the fifth to the second as the sixth to the fourth: prove that the first and fifth together will be to the second as the third and sixth together to the fourth.

15. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$.

16. If $a:b=c:d=e:f$, prove that $\dfrac{a^2ce-ac^2e+ace^2}{b^2df-bd^2f+bdf^2}=\dfrac{a^4}{b^4}$.

17. If $qr \propto s^2$, $rs \propto t^2$, and $st \propto r^2$, prove that $qs \propto r^2$.

18. There are two vessels each containing a mixture of wine and water in the ratios of 2 : 1 and 3 : 1 respectively. A mixture consisting of certain quantities from the two vessels, is composed of wine and water in the ratio of 5 : 2. Find the ratio of wine to water in a mixture consisting of the same quantities each taken out of the other vessel.

19. If $x \propto y^2$ and $y \propto z^{-\frac{1}{2}}$, and if when $x=7$, $y=5$, and when $y=4$, $z=25$: find x in terms of z.

20. Find two numbers whose difference, sum, and product are as the numbers 2, 3, and 5.

21. Find the duplicate ratio of 11 : 13, the sub-duplicate of 49 : 400, and that compounded of the ratio 2 : 3, 3 : 4, and 4 : 5.

22. Arrange in order of magnitude the ratios 3 : 7, 2 : 3, 5 : 8, and 11 : 21.

VI.—INDICES, SURDS, AND IMAGINARY QUANTITIES.

1. When the *index* is a *positive integer*,

$$\left.\begin{array}{l} x^m = x \,.\, x \,.\, x \ldots \text{ to } m \text{ factors by definition,} \\ \text{and } x^n = x \,.\, x \,.\, x \ldots \quad n \quad ,, \quad ,, \end{array}\right\}$$

Therefore $x^m \times x^n = (x \,.\, x \ldots \text{ to } m \text{ factors}) \times (x \,.\, x \ldots \text{ to } n \text{ factors})$

$$= x \,.\, x \ldots \text{ to } (m+n) \text{ factors} = x^{m+n}.$$

Similarly, if there be any number of *positive integers* $m, n, p, \ldots$ we get

$$x^m \,.\, x^n \,.\, x^p \ldots = x^{m+n+p+\cdots} \quad . \quad . \qquad \text{(i.)}$$

Thus $\left(\dfrac{z}{x}\right)^n = \dfrac{z}{x} \cdot \dfrac{z}{x} \ldots$ to n factors (by definition) $= \dfrac{z^n}{x^n} \ldots$ (ii.)

$$\text{and } \therefore \left(\frac{1}{x}\right)^n = \frac{1}{x^n}.$$

So also $(xyz \ldots)^n = (xyz \ldots)(xyz \ldots) \ldots$ to n factors $= x^n y^n z^n \ldots$

If m is $> n$, $x^m \div x^n = \dfrac{x \,.\, x \ldots \text{ to } m \text{ factors}}{x \,.\, x \ldots \text{ to } n \text{ factors}}$

$$= x \,.\, x \ldots \text{ to } (m-n) \text{ factors by cancelling } = x^{m-n}.$$

Next $(x^m)^n = x^m \,.\, x^m \ldots$ to n factors $= x \,.\, x \ldots$ to mn factors, for each x^m contains x repeated m times.

$$\therefore (x^m)^n = x^{mn} \quad . \quad . \quad . \quad . \qquad \text{(iii.)}$$

2. We will now extend the definition of x^m when m is no longer a positive integer, so that (i.) may still be true.

Thus $x^{\frac{p}{q}} x^{\frac{p}{q}} \ldots$ to q factors $= x^{\frac{p}{q}+\frac{p}{q}+\ldots \text{ to } q \text{ terms}} = x^{\frac{pq}{q}} = x^p$,

$$\text{and } x^{\frac{p}{q}} x^{\frac{p}{q}} \ldots \text{ to } q \text{ factors } = \left(x^{\frac{p}{q}}\right)^q \quad \therefore \left(x^{\frac{p}{q}}\right)^q = x^p,$$

That is $\therefore x^{\frac{p}{q}} = \sqrt[q]{x^p} =$ the q^{th} root of the p^{th} power of x. (iv.)

Similarly, $x^{\frac{1}{q}} x^{\frac{1}{q}} \ldots$ to p factors $= x^{\frac{1}{q}+\frac{1}{q}+\ldots \text{ to } p \text{ terms}} = x^{\frac{p}{q}}$

$$\therefore x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = \text{the } p^{th} \text{ power of the } q^{th} \text{ root of } x. \qquad \text{(v.)}$$

So also $x^{\frac{1}{q}} . x^{\frac{1}{q}} \ldots$ to q factors $= x^{\frac{1}{q}+\frac{1}{q}+\ldots \text{ to } q \text{ terms}} = x^{\frac{q}{q}} = x.$

$$\therefore \left(x^{\frac{1}{q}}\right)^q = x; \; \therefore x^{\frac{1}{q}} = \sqrt[q]{x} \quad . \quad . \qquad \text{(vi.)}$$

$$\text{And in the same way } \left(x^{\frac{p}{q}}\right)^r x = {}^{\frac{pr}{q}} \quad . \quad . \qquad \text{(vii.)}$$

[In results iv. . . . vii. p, q, r are *positive integers.*]

$$\text{Again, } x^m . x^0 = x^{m+0} = x^m, \; \therefore x^0 = 1 \quad . \quad . \qquad \text{(viii.)}$$

$$\text{and } x^{-m}. x^m = x^{-m+m} = x^0 = 1, \; \therefore x^{-m} = \frac{1}{x^m} \qquad \text{(ix.)}$$

3. To prove (α) that $\sqrt[q]{\frac{1}{x}} = \frac{1}{\sqrt[q]{x}}$, and ($\beta$) that $\left(\frac{1}{x}\right)^{\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}$, where p and q are *positive integers.*

(α.) $\left(\frac{1}{\sqrt[q]{x}}\right)^q = \frac{1}{x^{\frac{1}{q}}} \cdot \frac{1}{x^{\frac{1}{q}}} \ldots$ to q factors $= \frac{1}{x}$ by (vi.)

$$\therefore \sqrt[q]{\frac{1}{x}} = \frac{1}{\sqrt[q]{x}} \text{ or } \left(\frac{1}{x}\right)^{\frac{1}{q}} = \frac{1}{x^{\frac{1}{q}}}.$$

(β.) Therefore

$$\left(\frac{1}{x}\right)^{\frac{p}{q}} = \left\{\left(\frac{1}{x}\right)^p\right\}^{\frac{1}{q}} \text{ by (iv.) } = \left(\frac{1}{x^p}\right)^{\frac{1}{q}} \text{ by (ii.)} = \frac{1}{x^{\frac{p}{q}}} \text{ by } (\alpha.)$$

4. These extended definitions of § 2 may be shown to satisfy (iii.) also, in the following way:—

First, for *positive fractions:* let p, q, r, s be any *positive integers.*

Assume $\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}} = A$, then $A^s = \left\{\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}}\right\}^s = \left(x^{\frac{p}{q}}\right)^r$ by (iii.)

$$\therefore A^s = x^{\frac{pr}{q}} \text{ by (vi.)}$$

$$\therefore A = \sqrt[s]{x^{\frac{pr}{q}}} = x^{\frac{pr}{qs}} \text{ by (vii.)}$$

Next, for any quantities *integral* or *fractional*. Let m, n be any positive quantities.

$$\left.\begin{aligned} \text{Then } (x^m)^{-n} &= \frac{1}{(x^m)^n} = \frac{1}{x^{mn}} = x^{-mn} : \\ (x^{-m})^n &= \left(\frac{1}{x^m}\right)^n = \left(\frac{1}{x^{mn}}\right) \text{ [by § 3]} = x^{-mn} : \\ \text{and } (x^{-m})^{-n} &= \frac{1}{(x^{-m})^n} = \frac{1}{x^{-mn}} = x^{mn}. \end{aligned}\right\}$$

5. (i.) To prove that $(xy)^{\frac{p}{q}} = x^{\frac{p}{q}}\, y^{\frac{p}{q}}$ where p, q are positive integers.

Let $A = (xy)^{\frac{p}{q}}$, $\therefore A^q = (xy)^p = x^p y^p$ by § 1.

$$\therefore A^q = \left(x^{\frac{p}{q}}\right)^q \left(y^{\frac{p}{q}}\right)^q \text{ by (v.)} = \left(x^{\frac{p}{q}} y^{\frac{p}{q}}\right)^q \quad \therefore A = x^{\frac{p}{q}} y^{\frac{p}{q}}.$$

Similarly, $x^{\frac{p}{q}} \div y^{\frac{p}{q}} = \left(\dfrac{x}{y}\right)^{\frac{p}{q}}$

Combining this result with § 4, we see that universally

$$(xyz \ldots)^m = x^m y^m z^m \ldots \text{ whatever } m \text{ may be.}$$

(ii.) $x^{\frac{p}{q}} . y^{\frac{r}{s}} = x^{\frac{pr}{qr}} y^{\frac{pr}{ps}} = \left(x^{\frac{1}{qr}}\right)^{pr} \left(y^{\frac{1}{ps}}\right)^{pr} = \left(x^{\frac{1}{qr}} . y^{\frac{1}{ps}}\right)^{pr}.$

$$\textit{e.g. } \sqrt[4]{x^3}\,\sqrt[3]{y^4} = x^{\frac{3}{4}} y^{\frac{4}{3}} = \left(x^{\frac{1}{16}} y^{\frac{1}{9}}\right)^{12}.$$

(iii.) Let $x^{\frac{p}{q}} y^{\frac{r}{s}} = A$. Then $A^{qs} = \left(x^{\frac{p}{q}} y^{\frac{r}{s}}\right)^{qs}$

$$= \left(x^{\frac{p}{q}}\right)^{qs} \left(y^{\frac{r}{s}}\right)^{qs} = x^{ps} y^{qr}.$$

$$\therefore A = \sqrt[qs]{x^{ps} y^{qr}}, \text{ or } (x^{ps} y^{qr})^{\frac{1}{qs}}.$$

$$\textit{e.g. } x^{\frac{2}{3}} y^{\frac{3}{5}} = (x^{10} y^9)^{\frac{1}{15}} = \sqrt[15]{x^{10} y^9}.$$

6. *Definition.*—A root which cannot be exactly obtained is called a Surd or Irrational Quantity.

e.g. $\sqrt[2]{27}$ and $\sqrt[3]{4}$ are surds, but not $\sqrt[2]{4}$ and $\sqrt[3]{27}$, for the latter are equal to 2 and 3 respectively.

Similar surds are those which are or can be expressed with the same surd factors, the other factors being rational.

e.g. $3\sqrt{20}$ and $2\sqrt{245}$ are similar surds, for they can be expressed in the forms $6\sqrt{5}$ and $14\sqrt{5}$ by § 5.

Mixed surds consist of a rational and a surd factor.

$$e.g.\ 3a\sqrt{xz}.$$

Entire surds have no rational factor.

$$e.g.\ \sqrt{6bc}.$$

Mixed surds may always be expressed as entire surds.

$$e.g.\ 3a\sqrt{xz}=\sqrt{9a^2}\sqrt{xz}=\sqrt{9a^2xz} \text{ by § 5.}$$

7. *Examples:*

$$\sqrt[3]{81}=\sqrt[3]{27\times 3}=3\sqrt[3]{3}\ ;\ 2\sqrt[4]{7}=\sqrt[4]{16}\times\sqrt[4]{7}=\sqrt[4]{16\times 7}=\sqrt[4]{112}\ ;$$

$$\sqrt[3]{24a^3x+40a^3x^2}=\sqrt[3]{8a^3(3x+5x^2)}=2a\sqrt[3]{3x+5x^2}\ ;$$

$$5\sqrt{8}\times 3\sqrt{5}=15\sqrt{40}=30\sqrt{10}\ ;\ 8\sqrt[3]{56}\div 4\sqrt[3]{2}=2\sqrt[3]{28}\ ;$$

$$\sqrt{27}+\sqrt{48}=3\sqrt{3}+4\sqrt{3}=7\sqrt{3}\ ;$$

$$3\sqrt[3]{\tfrac{1}{4}}+5\sqrt[3]{\tfrac{1}{32}}=3\sqrt[3]{\tfrac{2}{8}}+5\sqrt[3]{\tfrac{2}{64}}=\tfrac{3}{2}\sqrt[3]{2}+\tfrac{5}{4}\sqrt[3]{2}=\tfrac{11}{4}\sqrt[3]{2}.$$

For proofs of methods used, see § 5.

8. The Square root of a Rational quantity cannot be partly rational and partly a quadratic Surd.

For if possible let $\sqrt{n}=a+\sqrt{m}$; then squaring and reducing we get $\sqrt{m}=\dfrac{n-a^2-m}{2a}$, which is contrary to the supposition that $\sqrt{m}$ is a surd.

Thus if $x+\sqrt{y}=a+\sqrt{b}$, then $\sqrt{y}=(a-x)+\sqrt{b}$, $\therefore a-x=0$,

$$\therefore x=a, \text{ and } \therefore \sqrt{y}=\sqrt{b}.$$

This is called *equating rational and irrational parts.*

9. (i.) If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Square both sides: thus $a+\sqrt{b}=x+2\sqrt{xy}+y$,

$$\therefore \sqrt{b}=2\sqrt{xy} \text{ by § 8, } \therefore a-\sqrt{b}=x-2\sqrt{xy}+y$$

$$\text{and } \therefore \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

(ii.) Similarly, if $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}$.

Raise both sides to the n^{th} power [see p. 81]: thus

$$a+\sqrt{b}=x^n+nx^{n-1}\sqrt{y}+\frac{n(n-1)}{\underline{|2}}x^{n-2}y+\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}y\sqrt{y}+\dots$$

$$\therefore a=x^n+\frac{n(n-1)}{\underline{|2}}x^{n-2}y+.\text{ and }\sqrt{b}=nx^{n-1}\sqrt{y}+\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}y\sqrt{y}+.$$

$$\therefore a-\sqrt{b}=x^n-nx^{n-1}\sqrt{y}+\frac{n(n-1)}{\underline{|2}}x^{n-2}y-\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}y\sqrt{y}\dots$$

Therefore $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}.$

10. To find the condition that $\sqrt{a+\sqrt{b}}$ may be expressed in the form $\sqrt{x}+\sqrt{y}$.

Square both sides: thus $a+\sqrt{b}=x+y+2\sqrt{xy}$,

$$\therefore x+y=a,\text{ and }2\sqrt{xy}=\sqrt{b},$$

$$\therefore (x-y)^2=(x+y)^2-4xy=a^2-b,\quad \therefore x-y=\sqrt{a^2-b},$$

and therefore a^2-b must be a perfect square.

11. (i.) To find, if possible, $\sqrt{a\pm\sqrt{b}}$ in the form $\sqrt{x}\pm\sqrt{y}$.

Take $d=\sqrt{a^2-b}$ so that $a^2-d^2=b$.

Then $\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+d}{2}}\pm\sqrt{\frac{a-d}{2}}$, as may easily be proved by squaring both sides.

e.g. To find $\sqrt{16+6\sqrt{7}}$.

Here $d=\sqrt{256-252}=2$,

$$\therefore \sqrt{16+6\sqrt{7}}=\sqrt{\frac{16+2}{2}}+\sqrt{\frac{16-2}{2}}=3+\sqrt{7}.$$

Another method:

Let $\sqrt{16+6\sqrt{7}}=\sqrt{x}+\sqrt{y}$,

$$\therefore 16+6\sqrt{7}=x+y+2\sqrt{xy},$$

$$\therefore x+y=16,\text{ and }2\sqrt{xy}=6\sqrt{7},$$

$$\therefore (x-y)^2=(x+y)^2-4xy=256-252=4,$$

$$\therefore x-y=2,\quad \therefore x=9\text{ and }y=7,$$

$$\therefore \sqrt{16+6\sqrt{7}}=\sqrt{9}+\sqrt{7}=3+\sqrt{7}.$$

(ii.) Sometimes a square root such as

$$\sqrt{10+2\sqrt{10}+2\sqrt{15}+2\sqrt{6}} \text{ may be found.}$$

Assume $\sqrt{10+2\sqrt{10}+2\sqrt{15}+2\sqrt{6}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Square both sides. Thus:

$$10+2\sqrt{10}+2\sqrt{15}+2\sqrt{6}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}.$$

Assume $2\sqrt{xy}=2\sqrt{10}$, $2\sqrt{xz}=2\sqrt{15}$, $2\sqrt{yz}=2\sqrt{6}$.

This gives $xy=10$, $xz=15$, $xz=6$.

$$\therefore x=5,\ y=2,\ z=3.$$

And these values also satisfy $x+y+z=10$.

Therefore required square root is $\sqrt{5}+\sqrt{2}+\sqrt{3}$.

(iii.) To find $\sqrt[3]{26+15\sqrt{3}}$.

Let $\sqrt[3]{26+15\sqrt{3}}=x+\sqrt{y}$,

Then by § 9 $\sqrt[3]{26-15\sqrt{3}}=x-\sqrt{y}$, $\therefore x^2-y=\sqrt[3]{(26)^2-(15\sqrt{3})^2}=1$,

Also $26+15\sqrt{3}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$, $\therefore 26=x^3+3xy$,

$\therefore x^3+3x(x^2-1)=26$, $\therefore 4x^3-3x=26$, $\therefore x=2$, and $\therefore y=3$,

and $\therefore \sqrt[3]{26+15\sqrt{3}}=2+\sqrt{3}$.

12. To rationalise the Denominator of a fraction.

(i.) $\dfrac{1+3\sqrt{5}}{2\sqrt{5}-\sqrt{2}}=\dfrac{(1+3\sqrt{5})(2\sqrt{5}+\sqrt{2})}{(2\sqrt{5}-\sqrt{2})(2\sqrt{5}+\sqrt{2})}=\dfrac{30+2\sqrt{5}+\sqrt{2}+3\sqrt{10}}{18}$;

(ii.) $\dfrac{1}{\sqrt{7}+\sqrt{3}+\sqrt{5}}=\dfrac{(\sqrt{7}+\sqrt{3})^2-5}{\sqrt{7}+\sqrt{3}-\sqrt{5}}=\dfrac{(\sqrt{7}+\sqrt{3}-\sqrt{5})(2\sqrt{21}-5)}{(2\sqrt{21}+5)(2\sqrt{21}-5)}$

$$=\frac{9\sqrt{3}+\sqrt{7}-2\sqrt{105}+5\sqrt{5}}{59}.$$

(iii.) Find the value of $\dfrac{2}{2-\sqrt{3}}$, having given that $\sqrt{3}=1{\cdot}7320508$

$$\frac{2}{2-\sqrt{3}}=\frac{2(2+\sqrt{3})}{4-3}=2\times 3{\cdot}7320508=7{\cdot}4641016.$$

13. To find a Factor which will rationalise $x^{\frac{a}{b}}\pm y^{\frac{c}{d}}$.

Take $\alpha=x^{\frac{a}{b}}$, $\beta=y^{\frac{c}{d}}$, and let n be the L.C.M. of b, d so that $\dfrac{na}{b}$ and $\dfrac{nc}{d}$ are *integral*, and therefore α^n and β^n are *rational*.

Then $\dfrac{a^n \pm \beta^n}{a \pm \beta} = a^{n-1} \pm a^{n-2}\beta + \text{etc.}$, which is the required factor.

The signs must be determined by I. §§ 10, p. 2.

Thus $a^n + \beta^n = (a+\beta)(a^{n-1} - a^{n-2}\beta + \ldots + \beta^{n-1})$ when n is *odd;*

$a^n - \beta^n = (a+\beta)(a^{n-1} - a^{n-2}\beta + \ldots - \beta^{n-1})$ when n is *even;*

$a^n - \beta^n = (a-\beta)(a^{n-1} + a^{n-2}\beta + \ldots + \beta^{n-1})$ when n is *odd* or *even.*

e.g. To find a rationalising factor for $\sqrt[2]{2^3} - \sqrt[3]{3^2}$.

Put $2^{\frac{3}{2}} = x$, $3^{\frac{2}{3}} = y$. Then $\dfrac{x^6 - y^6}{x+y} = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$,

$\therefore$ Factor is $2^{\frac{15}{2}} - 2^6 3^{\frac{2}{3}} + 2^{\frac{9}{2}} 3^{\frac{4}{3}} - 2^3 3^2 + 2^{\frac{3}{2}} 3^{\frac{8}{3}} - 3^{\frac{10}{3}}$. Ans.

15. *Definition.*—An imaginary expression is one which contains the square root of a negative quantity. It can be put in the form $a + \beta\sqrt{-1}$, where a, β are real quantities or expressions.

If $a + \beta\sqrt{-1} = \gamma + \delta\sqrt{-1}$, then will $a = \gamma$ and $\beta = \delta$. For, if not, let $\gamma = a + \epsilon$ then $a + \beta\sqrt{-1} = a + \epsilon + \delta\sqrt{-1}$.

$$\therefore \epsilon = (\beta - \delta)\sqrt{-1}, \quad \therefore \epsilon^2 = -(\beta - \delta)^2$$

i.e. the square of a real quantity is equal to a negative quantity, which is impossible;

$$\therefore a = \gamma, \text{ and therefore } \beta = \delta.$$

This is called *equating possible and impossible parts.* Cf. § 8.

From this it may be seen that $a + \beta\sqrt{-1} = 0$ only when both $a = 0$ and $\beta = 0$.

16. $a + \beta\sqrt{-1}$ and $a - \beta\sqrt{-1}$ are called *Conjugate* imaginary expressions; their product is $a^2 - (\beta\sqrt{-1})^2 = a^2 + \beta^2$, and $+\sqrt{a^2 + \beta^2}$ is called the *Modulus* of either of them; *i.e.* the modulus of either of two conjugate imaginary expressions is the positive square root of their product.

The modulus of the product of any two imaginary expressions $a + \beta\sqrt{-1}$ and $\gamma + \delta\sqrt{-1}$ = the product of their moduli.

For $(a + \beta\sqrt{-1})(\gamma + \delta\sqrt{-1}) = (a\gamma - \beta\delta) + (\beta\gamma + a\delta)\sqrt{-1}$,

$$\therefore \text{modulus of product} = \sqrt{(a\gamma - \beta\delta)^2 + (\beta\gamma + a\delta)^2}$$

$$\sqrt{a^2\gamma^2 + \beta^2\delta^2 + \beta^2\gamma^2 + a^2\delta^2} = \sqrt{a^2 + \beta^2} \times \sqrt{\gamma^2 + \delta^2} = \text{product of moduli.}$$

17. To express $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}$ in the form $A+B\sqrt{-1}$.

Multiply numerator and denominator by $c-d\sqrt{-1}$,

$$\therefore \frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{c^2-(d\sqrt{-1})^2}=\frac{ac+bd+bc\sqrt{-1}-ad\sqrt{-1}}{c^2+d^2},$$

$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\sqrt{-1}, \text{ which is of the required form.}$$

18. To find the square root of an imaginary expression.

Let $\sqrt{a+\beta\sqrt{-1}}=\sqrt{x}+\sqrt{y}\sqrt{-1}$.

Square both sides thus: $a+\beta\sqrt{-1}=x-y+2\sqrt{xy}\sqrt{-1}$

$\because$ by § 15. $x-y=a$, $2\sqrt{xy}=\beta$, $\therefore (x+y)^2=(x-y)^2+4xy=a^2+\beta^2$

$$\therefore x=\frac{a+\sqrt{a^2+\beta^2}}{2}, \text{ and } y=\frac{a-\sqrt{a^2+\beta^2}}{2},$$

$$\therefore \sqrt{a+\beta\sqrt{-1}}=\left(\frac{\sqrt{a^2+\beta^2}+a}{2}\right)^{\frac{1}{2}}+\left(\frac{\sqrt{a^2+\beta^2}-a}{2}\right)^{\frac{1}{2}}\sqrt{-1}.$$

Ex. Find $\sqrt[4]{-64a^4}$ in the form $a+\beta\sqrt{-1}$.

$$\sqrt[4]{-64a^4}=\pm 2a\sqrt[4]{-4}=\pm 2a\sqrt[2]{-2\sqrt{-1}}$$

$$\text{Let } \sqrt{-2\sqrt{-1}}=\sqrt{x}+\sqrt{y}\sqrt{-1}$$

$$\text{Then } x-y=0, \text{ and } 2\sqrt{xy}=-2, \ \therefore \sqrt{x}=1 \text{ and } \sqrt{y}=-1$$

$$\therefore \sqrt[4]{-64a^4}=\pm 2a(1-\sqrt{-1}).$$

EXAMPLES. VI.

1. Multiply $a^{3m-n}+a^{2m}+a^{4m-2n}$ by a^m-a^n.

2. Subtract $\dfrac{x^n}{(x+y)^m}$ from $\dfrac{x^{n-1}}{(x+y)^{m-1}}$.

3. Simplify $\dfrac{b}{a^{m-n}}+\dfrac{c}{a^{m-p}}+\dfrac{d}{a^{m-q}}-\dfrac{e}{a^m}$.

4. Divide $\dfrac{x^{m+3n}y^{7m-8n}}{x^{4m-7n}y^{3m-11n}}$ by $\dfrac{x^{2m-6n}y^{5m+6n}}{x^{5m-17n}y^{2m+4n}}$.

5. Subtract $\dfrac{a^m - b^m}{a^m + b^m}$ from $\dfrac{a^m + b^m}{a^m - b^m}$.

6. Simplify $\sqrt[3]{a^6 b^{\frac{3}{2}} c^{\frac{1}{2}}} \times b^{\frac{1}{2}} \times (c^{\frac{1}{6}} a^2)^{-\frac{1}{2}}$.

7. If $a = xy^{p-1}$, $\beta = xy^{q-1}$ and $\gamma = xy^{r-1}$, prove that $a^{q-r}\beta^{r-p}\gamma^{p-q} = 1$.

8. Multiply $x^{\frac{2}{3}} + 2y^{\frac{1}{2}} + 3z^{\frac{3}{4}}$ by $x^{\frac{2}{3}} - 2y^{\frac{1}{2}} - 3z^{\frac{3}{4}}$.

9. Simplify $\dfrac{a^{2n} + a^n b + b^2}{a^{4n} - a^{3n}b + a^{2n}b^2 - a^n b^3 + b^4} - \dfrac{1}{a^{2n} - a^n b + b^2}$.

10. Reduce $\dfrac{a + b + c + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}}$.

11. Simplify the following : (a.) $\dfrac{(2ab)^5(3ab)^2(5a)^4}{(3b)^3(4ab)^6}$;

(β.) $\left(\dfrac{a^9 b^{28} c^{47}}{d^{10} e^{29}}\right)^{17} \times \left(\dfrac{d^9 e^{26}}{a^b b^{25} c^{42}}\right)^{19}$; ($\gamma$.) $\left(\dfrac{2ab}{3cd}\right)^{-3} \left(\dfrac{4cd}{5ab}\right)^{-2} \left(\dfrac{5ab}{2cd}\right)^{-4}$.

12. Divide $(5a^2 + 8ab - 21b^2)^n$ by $(a + 3b)^n$.

13. Multiply $x^m + x^{m+1}y^{-1} + x^{m+2}y^{-2} + x^{m+3}y^{-3}$ by $x^2 - 2x^3y^{-1} + x^4y^{-2}$.

14. Reduce $(3^{\frac{5}{3}} - 2^{\frac{1}{2}}3^{\frac{4}{3}} + 6 - 2^{\frac{3}{2}}3^{\frac{2}{3}} + 2^2 3^{\frac{1}{3}} - 2^{\frac{5}{2}}) \times (3^{\frac{1}{3}} + 2^{\frac{1}{2}})$.

15. Divide $15\,(a - b) + 16\sqrt{ab}$ by $3\sqrt{a} + 5\sqrt{b}$.

16. Simplify $\dfrac{2y^{-\frac{3}{2}} - 2y^{-\frac{1}{2}} + 1 - y^{\frac{1}{2}}}{3 - 3y^{\frac{1}{2}} + 4y^{\frac{3}{2}} - 4y}$.

17. Extract the square root of

$$\frac{1}{4}x^{\frac{2}{3}}a^{\frac{2}{3}} + \sqrt[3]{\frac{a^4}{x^2}} - a^{\frac{1}{3}} + \frac{2}{3}\frac{a^{\frac{2}{3}}}{x^{\frac{1}{3}}} - \frac{1}{3}x^{\frac{1}{3}}a^{-\frac{1}{3}} + \frac{1}{9}.$$

18. Simplify

$\sqrt{9m + 25n - 30\sqrt{mn}}$; $\sqrt{2p^2 + q^2 + 2p\sqrt{p^2 + q^2}}$; and $\sqrt{2p + 2\sqrt{p^2 - q^2}}$.

19. Prove that $\dfrac{x^{\frac{3}{2}} + y^{\frac{3}{2}} + xy(x^{-\frac{1}{2}} + y^{-\frac{1}{2}})}{x^{\frac{3}{2}} - y^{\frac{3}{2}} - xy(x^{-\frac{1}{2}} - y^{-\frac{1}{2}})} = \dfrac{x + y}{x - y}$.

20. Find the value of $27x + 48x^2 - 8x^4$ when $x = \dfrac{1}{4}(\sqrt{21} - 3)$.

21. Show that the result is the same whether $2 + \sqrt{3}$ or $2 - \sqrt{3}$ be substituted for x in the expression $x^3 - 15x + 5$.

22. Divide $1+x^6$ by $1-\sqrt{3}x+x^2$.

23. Find the continued product of

$1+2\sqrt{2}$, $4-\sqrt{3}$, $\sqrt{2}+\sqrt{3}$, $4+\sqrt{3}$, $2\sqrt{2}-1$, and $\sqrt{3}-\sqrt{2}$.

24. Find the square root of $57-12\sqrt{15}$, and of $31+\sqrt{600}$.

25. If $\beta=\sqrt{\dfrac{1-e}{1+e}}$, prove that $\dfrac{1-\beta}{1+\beta}=\dfrac{e}{1+\sqrt{1-e^2}}$.

26. Prove that $\dfrac{3}{2}(\sqrt{3}+1)^2-2(\sqrt{2}+1)^2=\sqrt{59-24\sqrt{6}}$.

27. Rationalise the denominators of

$$\frac{3+\sqrt{2}}{\sqrt{2}-1+\sqrt{3}},\ \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} \text{ and } \frac{1+\sqrt{3}}{2\sqrt{2}-3\sqrt{3}}.$$

28. Add together $\dfrac{1}{4(1+\sqrt{x})}$, $\dfrac{1}{4(1-\sqrt{x})}$ and $\dfrac{1}{2(1+x)}$.

29. Prove that $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}=\sqrt{10}$; and find the square root of $49-20\sqrt{6}$.

30. Find the square root of $\dfrac{9}{8}-\sqrt{\dfrac{9}{8}}$ and of $100-2\sqrt{2499}$.

31. Find the value of

$$x^3-x^2+3x+5 \text{ when } x=1+2\sqrt{-1}.$$

32. Divide $x^4+a^2x^2+a^4$ by $x-\sqrt{-3ax}-a$.

33. Simplify $\sqrt{17+20\sqrt{-2}}+\sqrt{17-20\sqrt{-2}}$.

34. Reduce $\dfrac{(a+b\sqrt{-1})^2}{a-b\sqrt{-1}}-\dfrac{(a-b\sqrt{-1})^2}{a+b\sqrt{-1}}$ to a single fraction with a rational denominator.

35. Find the value of $x^3+3a^2x+4a^3$ when $\dfrac{2x}{a}=1+\sqrt{-15}$.

36. Find the fourth power of $-\sqrt{-2\sqrt{-3}}$.

37. Add together $\dfrac{1}{x-1}$, $\dfrac{2}{2x+1+\sqrt{-3}}$ and $\dfrac{2}{2x+1-\sqrt{-3}}$.

38. Prove that $\sqrt{7+\sqrt{-15}}+\sqrt{7-\sqrt{-15}}=\sqrt{30}$.

39. Express the values of $(\sqrt{-1})^m$ where m has each of the four forms $4n$, $4n+1$, $4n+2$, $4n+3$; n being a positive integer.

40. Find the value of $(x-10)^2+(x-6)^2$ when $x=8+2\sqrt{-1}$.

41. Simplify (i.) $\dfrac{3^{\frac{2}{3}}+3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}+1}+\dfrac{3^{\frac{2}{3}}-3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}-1}$; and (ii.) $\dfrac{\dfrac{1-\sqrt{2}}{1+\sqrt{2}}}{\dfrac{3-2\sqrt{2}}{3+2\sqrt{2}}}$.

42. Multiply $\sqrt{2x}+\sqrt{2(2x-1)}-\dfrac{1}{\sqrt{2x}}$ by $\dfrac{1}{\sqrt{2x}}+\sqrt{2(2x-1)}-\sqrt{2x}$.

43. Simplify—

$$\frac{(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{(\sqrt{2}+\sqrt{3}+\sqrt{5})^2};\ \text{and } (3+\sqrt{2})^3+(3-\sqrt{2})^3.$$

44. Find the square root of $16+6\sqrt{7}$.

45. Find the value of $\dfrac{\sqrt{3-\sqrt{5}}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}$ correct to three places of decimals.

46. Express $\dfrac{5^{\frac{1}{4}}-7^{\frac{1}{4}}}{5^{\frac{1}{4}}+7^{\frac{1}{4}}}$ by an equivalent fraction with a rational denominator.

47. Find a factor which will rationalise $3^{\frac{1}{3}}+2^{\frac{1}{2}}$, and obtain the numerical value of the product.

48. Simplify $\dfrac{92}{\sqrt{32}-3}\times\dfrac{68}{\sqrt{18}+1}\times\dfrac{6-\sqrt{20}}{16}$.

49. Find the value of x^2-6x+7 when $x=3-\sqrt{3}$.

50. Find the continued product of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, $\sqrt{b}+\sqrt{c}-\sqrt{a}$, $\sqrt{c}+\sqrt{a}-\sqrt{b}$, and $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

51. Find the square root of $-7-24\sqrt{-1}$, and prove the result equal to $\{1+2\sqrt{-1}\}^2$.

52. Find square roots of $280+56\sqrt{21}$, and $43+12\sqrt{7}$.

53. Reduce to its simplest form—

$$(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}}.$$

54. Find the square root of $2+2\sqrt{1-a^2}$.

55. Find the expression whose square is $3x-1+2\sqrt{2x^2+x-6}$.

56. Find the square root of $2\sqrt{-1}$ in the form $\alpha+\beta\sqrt{-1}$.

57. Simplify $[(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}][(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}+(a^2-b^2)^{\frac{1}{3}}]$.

58. Distinguish a^3, $3a$, $\dfrac{a}{3}$, $a^{\frac{1}{3}}$, $\dfrac{3}{a}$, a^{-3}; when $a=8$.

59. Extract the square root of

$$ab-d^2+4c^2\pm2\sqrt{4abc^2-abd^2}.$$

60. Find the 8^{th} root of 1679616; the 6^{th} root of 4826809; and the 9^{th} root of 134217728.

61. Prove that

$$\frac{1}{\sqrt{11-2\sqrt{30}}}-\frac{3}{\sqrt{7-2\sqrt{10}}}-\frac{4}{\sqrt{8+4\sqrt{3}}}=0.$$

62. Multiply $a^{\frac{3}{2}}x^{-\frac{3}{2}}+a^{-\frac{3}{2}}x^{\frac{3}{2}}+a^{\frac{1}{2}}x^{-\frac{1}{2}}+a^{-\frac{1}{2}}x^{\frac{1}{2}}$, by $a^{\frac{1}{2}}x^{-\frac{1}{2}}-1+a^{-\frac{1}{2}}x^{\frac{1}{2}}$, and prove the truth of your result by division.

63. Find the square root of $x^4-2\sqrt{2}x^3+2(\sqrt{3}+1)x^2-2\sqrt{6}x+3$.

64. Find the cube root of $50+19\sqrt{7}$.

65. Multiply $x^{\frac{1}{2}}+1+x^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}}-1+x^{\frac{1}{2}}$; and divide

$$x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}} \text{ by } x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2.$$

66. If $x=\dfrac{\sqrt{5}-1}{4}$, prove that $8x\sqrt{1-x^2}=\sqrt{10-2\sqrt{5}}$.

67. Find the square root of $2\{1+\sqrt{1+x^6}\}+x^4$.

68. Rationalise the denominator of $\dfrac{3\sqrt{3}+2\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}$.

69. Simplify $\dfrac{5+2\sqrt{-3}}{2-\sqrt{-3}}+\dfrac{2}{2+\sqrt{-3}}-\dfrac{4}{1-\sqrt{-3}}$.

70. If $\sqrt{2}=1{\cdot}414$ find the value of $\dfrac{1}{3-\sqrt{2}}+\dfrac{9}{\sqrt{2}}-\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$.

71. Find the square root of (i.) $12+2\sqrt{6}+2\sqrt{14}+2\sqrt{21}$ and of (ii.) $22+6\sqrt{2}+6\sqrt{11}+2\sqrt{22}$.

72. Find the cube root of $72+32\sqrt{5}$.

73. Prove that the product of two *dissimilar* quadratic surds cannot be a rational quantity.

74. Prove that a quadratic surd cannot be equal to the sum or the difference of two *dissimilar* quadratic surds.

75. Simplify $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}+\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$.

VII.—PROGRESSIONS.

1. An *Arithmetical Progression* is a series of terms each of which is equal to the preceding term $+$ a constant quantity, called the *common difference*, which may be either positive or negative.

In a given A. P. let $a=$ 1st term; $d=$ *common difference* $=$ any term $-$ preceding term; $n=$ number of terms; $l=n^{th}$ or last term $=a+(n-1)d$; $S=$ sum of first n terms.

Thus $S=a+(a+d)+(a+2d)+\ \ldots\ +l$

and $S=l+(l-d)+(l-2d)+\ \ldots\ +a$ by reversion of the series.

$\therefore$ by addition $2S=(a+l)+(a+l)+(a+l)+\ldots+(a+l)=n(a+l)$

$$\therefore S=\frac{n}{2}(a+l)=\frac{n}{2}\{2a+(n-1)d\}.$$

2. To insert m Arithmetic Means between p and q, that is, to find an A. P. of $m+2$ terms, of which p and q are the 1st and $(m+2)^{\text{th}}$ terms respectively.

Let d be the common difference of the required A. P.

Then $q=p+(\overline{m+2}-1)d$, $\therefore\ d=\dfrac{q-p}{m+1}$ and the *Means* are

$$p+d,\ p+2d,\ p+3d,\ \ldots\ p+(m-1)d.$$

3. A *Geometrical Progression* is a series of terms each of which is equal to the preceding term $\times$ a constant quantity, called the

common ratio, which may be either positive or negative, and either $>$ or <1.

In a given G. P. let $a=$1st term; $r=$*common ratio*$=$any term$\div$ preceding term; $n=$number of terms; $l=n^{th}$ or last term$=ar^{n-1}$; $S=$sum of first n terms.

$$\text{Thus } S=a+ar+\ \dots\ +ar^{n-2}+ar^{n-1},$$

$$\therefore rS=\quad ar+ar^2\ +\ \dots\ +ar^{n-1}+ar^n,$$

$$\therefore \text{by subtraction } (1-r)S=a\qquad\qquad -ar^n=a(1-r^n)$$

$$\therefore S=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1}=\frac{rl-a}{r-1}$$

If r is <1 and $n=\infty$, then $r^n=0$

$$\text{and } S \text{ (i.e. \textit{the Sum to infinity})} =\frac{a}{1-r}.$$

4. To insert m Geometric Means between p and q, that is, to find a G. P. of $m+2$ terms, of which p and q are the 1st and $(m+2)^{\text{th}}$ terms respectively.

Let r be the required common ratio. Then

$$q=pr^{\overline{m+2}-1},\ \therefore \frac{q}{p}=r^{m+1},\ \therefore r=\sqrt[m+1]{\frac{q}{p}};\ \text{and the \textit{Means} are}$$

$$pr,\ pr^2,\dots pr^{m-1}.$$

5. *An Harmonical Progression* is a series of terms the reciprocals of which are in Arithmetical Progression.

e.g. $a, b, c, d\dots$ are in H. P. if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\dots$ are in A. P.

To insert means or to continue a series either way, invert and proceed as for A. P. and then invert back again.

There is no formula in this case for S.

6. If three quantities a, b, c are in H. P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

$$\therefore \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},\ \therefore \frac{a-b}{ab}=\frac{b-c}{bc};$$

$$\therefore \frac{a}{c}=\frac{a-b}{b-c},\ \therefore a:c::a-b:b-c.$$

In words this result is; three quantities are in H. P. if the first is to the third as the first minus the second is to the second minus the third.

7. Let A, G, H be respectively the Arithmetic Mean, the Geometric Mean or Mean Proportional, and the Harmonic Mean between any two quantities a and b, so that a, A, b are in A. P., a, G, b are in G. P., and a, H, b are in H. P. Then by the Definitions

$$A - a = b - A \quad \therefore A = \frac{a+b}{2}:$$

$$\frac{G}{a} = \frac{b}{G}, \therefore G^2 = ab, \therefore G = \sqrt{ab}:$$

$$\text{and } \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}, \therefore \frac{2}{H} = \frac{1}{a} + \frac{1}{b}, \therefore H = \frac{2ab}{a+b}.$$

Therefore $A \,.\, H = \dfrac{a+b}{2} \,.\, \dfrac{2ab}{a+b} = ab = G^2$.

i.e. The Geometric Mean between any two quantities is also the Geometric Mean between the Arithmetic and Harmonic Means of the same two quantities.

8. *Examples.* (i.) Find five numbers in Arithmetical Progression, such that their sum $=55$, and the sum of their squares $=765$.

Let the numbers be $x-2y, x-y, x, x+y, x+2y$,

$$\therefore x-2y+x-y+x+x+y+x+2y=55, \therefore 5x=55, \therefore x=11,$$

$$\therefore (11-2y)^2+(11-y)^2+11^2+(11+y)^2+(11+2y)^2=765.$$

$$\therefore 121-44y+4y^2+121-22y+y^2+121+121+22y+y^2+121+44y + 4y^2=765,$$

$$\therefore 10y^2=765-605=160, \therefore y^2=16, \therefore y=\pm 4.$$

Therefore the numbers are 3, 7, 11, 15, and 19. Ans.

(ii.) If a, b, c, d be any four consecutive terms of an A. P. prove that $bc-ad$ is positive.

Let the terms be $x-3y, x-y, x+y, x+3y$;

then $bc-ad=(x-y)(x+y)-(x-3y)(x+3y)=x^2-y^2-(x^2-9y^2)$
$=8y^2$, which is necessarily *positive* being a *square*.

[Notice that x in these two examples is taken as the middle of the A. P. whether there be a term there or not.]

(iii.) To find the sum of the first n natural numbers.

$$S = 1+2+3+\ \dots\ +n$$
$$= n+\ \dots\dots\ +1$$
$$\therefore\ S = \frac{n(n+1)}{2}.$$

(iv.) To find the sum of the first n *odd* natural numbers.

$$S = 1+3+5+\ \dots\ +(2n-1)$$
$$= \frac{n}{2}\{2+(n-1)2\} = n^2.$$

EXAMPLES. VII.

1. Sum $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\dots$ and $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\dots$ each to 7 terms.

2. The first term of an A. P. is n^2-n+1, the common difference 2, and the number of terms n. Find the sum.

3. Sum $1{\cdot}3+3{\cdot}5+5{\cdot}7+\dots$ to n terms.

4. There are n Arithmetic Means between 1 and 31 and the 7^{th} : the $(n-1)^{\text{th}}$: : 5 : 9. Find n.

5. Sum $5+20+80+\dots$ and $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\dots$ each to 8 terms.

6. Sum $1-\frac{2}{5}+\frac{4}{25}-\dots$ and $\frac{a+x}{a-x}+\frac{a-x}{a+x}+\left(\frac{a-x}{a+x}\right)^3+\dots$ each to ∞.

7. Continue the H. P. 2, 3, 6 to 3 terms both ways.

8. Insert 6 Harmonic Means between 3 and $\frac{6}{23}$.

9. Find a fourth harmonical proportional to 6, 8, and 12.

10. Sum the following series :—

(i.) $2+5+8+11+\dots$ to n terms.

(ii.) $2-5-12-\dots$ to n terms.

(iii.) $8+4+2+\dots$ to 15 terms.

(iv.) $\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\dots$ to ∞.

(v.) $\frac{5}{6}+\frac{1}{2}+\frac{1}{6}+\ldots$ to 17 terms.

(vi.) $2\frac{1}{2}+3\frac{1}{6}+4\frac{1}{18}+\ldots$ to 8 terms.

11. If a, b, na are in A. P., prove that a, b, $\frac{n+1}{2}b$ are in G. P.

12. Find (S) the sum of $2n$ terms of $a+(a-b)+(a-2b)+\ldots$ What is the least number of terms reckoned from a that will make S negative? What is the value of S when $b=\frac{a}{n}$?

13. Show that the product of any odd number of consecutive terms of a G. P. will be equal to the n^{th} power of the middle term, n being the number of terms.

14. The first 2 terms of an infinite G. P. are together equal to 1, and every term is twice the sum of all the terms that follow. Find the series.

15. In any G. P. if the sum of the first $2n$ terms be p times that of the first n, and the sum of $4n$ be q times that of $2n$, then $(p-1)^2=q-1$.

16. The first term of a G. P., whose common ratio is $\frac{1}{2}$, is 2^n, show that the sum of all the terms after the n^{th} is 2.

17. If $\frac{1}{a}+\frac{1}{c}=\frac{1}{b-a}+\frac{1}{b-c}$, prove that a, b, c are in H. P.

18. If the m^{th} term of an H. P. $=n$, and the $n^{th}=m$, prove that the $(m+n)^{th}$ term $=\frac{mn}{m+n}$.

19. A man borrows every year £25, upon which he pays interest at the rate of 4 per cent. per annum. In how long a time will the interest that he has paid amount to £91?

20. The A. M. between a and $b=(1+x^2)^2$ and the H. M. $=(1-x^2)^2$. Find the G. M., and determine also a and b.

21. The common difference of 4 numbers in A. P. is 1, and their product 120. Find them.

22. Find 6 numbers in A. P. such that their sum $=48$ and the sum of their squares $=454$.

23. The sum of 7 numbers in A. P. is 28, and the sum of their cubes is 784. Find the series.

24. Find 4 numbers in A. P. such that, if 2, 4, 8, 15 be added to them respectively, the sums shall be in G. P.

25. If a, b, c, d be 4 consecutive terms of an A. P., show that $bc-ad$ must be positive.

26. The sum of 3 numbers in H. P. is 11, and the sum of their squares is 49. Find them.

27. If 4 quantities are in A. P. *or* in G. P., prove that the sum *or* the product of the extremes = the sum *or* the product of the means.

28. Prove that $bc-a^2$, $ca-b^2$, and $ab-c^2$ are in A. P., if a, b, c are in A. P.

29. If the $(n-1)^{th}$ and $(n+1)^{th}$ terms of a G. P. are the A. M. and H. M. between any two quantities, prove that the n^{th} term is the G. M. between the same.

30. Find the number of terms of the series $24+21+18+\ldots$ of which 78 is the sum, and explain the double answer.

31. If a, b, c are in H. P., and x is the H. Mean between a and b, and y the H. Mean between b and c, prove that x, b, y are in H. P.

32. If a, b, and c are in harmonical progression, show that

$$\left[\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right]\left[\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right]=\frac{4}{ac}-\frac{3}{b^4}.$$

33. Between two quantities A and B a harmonic mean H is inserted. Between A and H and between H and B geometric means G_1 and G_2 are inserted, and it is found that G_1, H, G_2 are in A. P. Find the ratio of A to B.

34. Insert 9 Arithmetic Means between 9 and 109; and if 9 be the second term of the series find the 500th.

35. If $S_1=a^x-a^{-x}$, $S_2=a^{2x}-a^{-2x}$, . . . etc., prove that

$$S_1+S_3+\ldots+S_{2n-1}=\frac{(S_n)^2}{S_1}.$$

36. Sum the series

$$\text{(i.) } \frac{2}{3}+3+\frac{16}{3}+\frac{23}{3}+\ \ldots \text{ to 16 terms.}$$

$$\text{and (ii.) } \frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ \ldots \text{ to } n \text{ terms.}$$

37. Find the sum of 8, $7\frac{1}{3}$, $6\frac{2}{3}$ . . . to 19 terms; and find how many terms of the series 1, 3, 5, 7 . . . must be added together to make the 6th power of 12.

38. To each of three consecutive terms of a geometrical series the second of the three is added. Show that the three resulting quantities are in harmonical progression.

39. Find the sums of the series—

$$\text{(i.) } 1+\frac{7}{6}+\frac{4}{3}+\frac{3}{2}+\ \ldots \text{ to 25 terms,}$$

$$\text{and (ii.) } 1+\frac{1}{6}+\frac{1}{36}+\frac{1}{216}+\ \ldots \text{ to infinity.}$$

40. Find the ratio of two numbers the arithmetic mean of which is $\frac{5}{4}$ of the geometric mean.

41. Fifty stones are placed in a straight line upon the ground at the distance of one yard from one another. How far will a person walk in bringing them one by one to a basket which is placed one yard from the first stone?

42. Find 3 numbers in A. P. such that their sum$=15$, and the sum of their cubes$=495$.

43. If $a+b+c=2s$, and if $s-a$ be the geometric mean between $s-b$ and $s-c$, prove that $2(s-a)$ will be the harmonic mean between b and c.

44. Find the sum of $2n+1$ terms of the series a, $a+d$, $a+2d$, . . . beginning with the $(n+1)^{th}$ term.

45. If a, b, c, d, e are in G. P. prove that $c(a+2c+e)=(b+d)^2$.

46. Find 4 numbers in A. P. such that their common difference is 3 and their product 280.

47. There are p arithmetical progressions each beginning from unity, the common differences are 1, 2, 3 . . . p; show that the sum of their n^{th} terms $=\dfrac{(n-1)p^2+(n+1)p}{2}$.

48. The sum of 4 numbers in A. P. is 56 and the sum of their squares is 864. Find them.

49. Find the 28th term of the series 13, $12\frac{2}{3}$, $12\frac{1}{3}$, etc.

50. Sum the series $2-\dfrac{1}{3}+\dfrac{1}{18}-\dfrac{1}{108}+\ldots$ to ∞.

51. Sum to ∞ the series

$$\frac{1}{\sqrt{2}(1+\sqrt{2})}+\frac{1}{(1+\sqrt{2})(2+\sqrt{2})}+\frac{1}{(2+\sqrt{2})(3+2\sqrt{2})}+\text{etc.}$$

52. The difference between two numbers is 48, and their arithmetic mean exceeds their geometric by 18. Find the numbers.

53. Prove that $\dfrac{1}{3}+\dfrac{1}{6\sqrt{-1}}-\dfrac{1}{12}-\ldots$ to $\infty=\dfrac{4-2\sqrt{-1}}{15}$.

54. Sum $\dfrac{1}{3}-\dfrac{1}{3\,.\,2}+\dfrac{1}{3\,.\,2^2}-\dfrac{1}{3\,.\,2^3}+\ldots$ to ∞.

55. Find three whole numbers in A. P. such that the square of the least added to the product of the two greater may make 28, but the square of the greatest added to the product of the two less may make 44.

56. The sum of the first two terms of an A. P. is 18 and of the next three terms is 12. How many terms must be taken to make 128?

57. If 3 quantities are in an increasing A. P., show that the second has to the first a greater ratio than the third to the second.

58. In any H. P. the product of the first two terms is to the product of any two adjacent terms as the difference between the first two is to the difference between the other two.

59. The sum of n arithmetic means between 1 and 19 is to the sum of the first $n-2$ of them as 5 : 3. Find the means.

60. If 4 positive quantities be in G. P. the sum of the two extremes is greater than the sum of the two means.

VIII.—PERMUTATIONS, COMBINATIONS, AND PROBABILITIES.

1. *Definitions.*

(i.) By the number of *Permutations* of n things r at a time is meant the number of differently arranged sets, each containing r things which can be formed out of the n things. It is represented by the symbol ${}_nP_r$.

(ii.) By the number of *Combinations* is meant the number of different sets of r things which may be selected out of the n things, without regard to the arrangement of the r things in each set. It is represented by the symbol ${}_nC_r$.

2. To find ${}_nP_r$.

Let the n things be denoted by the letters $a, b, c, d. \ldots$ Omit one of the letters, say a, and form the remaining $n-1$ letters into permutations taken $r-1$ at a time; their number will be represented by the symbol ${}_{n-1}P_{r-1}$. Then before each of these permutations place a; we thus get ${}_{n-1}P_{r-1}$ permutations of n things taken r together in each of which a stands first; and if in the same way each of the n letters $a, b, c, d \ldots$ be put first in turn, we shall get all the possible permutations. Therefore

$$
\begin{aligned}
{}_nP_r &= n\,{}_{n-1}P_{r-1}.\\
\text{Similarly, } {}_{n-1}P_{r-1} &= (n-1)\,{}_{n-2}P_{r-2}\\
{}_{n-2}P_{r-2} &= (n-2)\,{}_{n-3}P_{r-3}\\
\ldots &= \ldots\ldots\ldots\\
{}_{n-r+2}P_2 &= (n-r+2)\,{}_{n-r+1}P_1\\
&= (n-r+2)(n-r+1).
\end{aligned}
$$

For ${}_{n-r+1}P_1$, *i.e.* number of permutations of $n-r+1$ things 1 at a time, clearly $=n-r+1$.

Multiply together all the equations and cancel out like factors.

Thus ${}_nP_r=n(n-1)(n-2) \ldots (n-r+1)$.

[Notice that there are r factors on the right-hand side.]

Hence ${}_nP_n = n(n-1)(n-2) \ldots 2 \,.\, 1$, which is written $\underline{|n}$, and read *factorial n.*

3. To find ${}_nC_r$.

(i.) Each combination of r things produces by itself ${}_rP_r = \underline{|r}$ permutations.

$$\text{Hence } {}_nC_r = \frac{{}_nP_r}{\underline{|r}} = \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

$$\text{Thus } {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}} = {}_nC_r.$$

(ii.) This value of ${}_nC_r$ may also be obtained directly in the same way as that of ${}_nP_r$ in § 2, as follows.

The number of combinations $r-1$ at a time which can be formed out of the $n-1$ things $b, c, d \ldots k$ is denoted by ${}_{n-1}C_{r-1}$. Replace a. We thus get ${}_{n-1}C_{r-1}$ combination of n things r at a time which contain one particular thing a. The same may be said of each of the n things. But in this way every combination would occur r times according as we began by omitting in turn each of the r things contained in it.

$$\therefore {}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1}.$$

$$\text{Similarly, } {}_{n-1}C_{r-1} = \frac{n-1}{r-1}\,{}_{n-2}C_{r-2}$$

$$\ldots = \ldots\ldots$$

$${}_{n-r+2}C_2 = \frac{n-r+2}{r-r+2}\,{}_{n-r+1}C_1 = \frac{n-r+2}{2}(n-r+1).$$

Multiply together all the equations, and cancel out like factors.

$$\text{Thus } {}_nC_r = \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}.$$

4. The number of Permutations of n things taken *all* at a time, of which p are of one kind, q of another, etc., is $\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\ldots}$

For, let N = the required number. Then if we change the p like things and make them all different, we can change their positions in $\underline{|p}$ ways among themselves: thus we should multiply N by $\underline{|p}$. Similarly for all

the other like things. Thus eventually we should make all the n things different, and should get

$$N \times \underline{|p} \times \underline{|q} \times \ldots = {}_nP_n = \underline{|n}$$

$$\therefore N = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\ldots}.$$

e.g. The number of different words which can be made out of the letters of *Mississippi*

$$= \frac{\underline{|11}}{\underline{|4}\,\underline{|4}\,\underline{|2}} = \frac{11.10.9.8.7.6.5.4.3.2}{4.3.2.4.3.2.2} = 34650.$$

5. Complementary Combinations.

Whenever we take a set of r things out of n we leave behind a corresponding set of $n-r$ things; such sets are called *complementary*. Hence it is clear that ${}_nC_r = {}_nC_{n-r}$, as was proved otherwise in § 3.

6. To find for what value of r, ${}_nC_r$ is greatest.

We have

$${}_nC_2 = \frac{n(n-1)}{1.2},\ {}_nC_3 = \frac{n(n+1)(n-2)}{1.2.3}, \text{etc.} \ldots {}_nC_r = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r},$$

therefore the value of ${}_nC_r$, as we give to r successive integral values, will begin to decrease after that the last introduced factor in the numerator is the number next greater than or equal to the last factor in the denominator.

(i.) Let n be *odd*. Then $\because$ the sum of the two last factors always $= n+1$, $\therefore$ when ${}_nC_r = \dfrac{n(n-1)\ldots\left(\frac{n+1}{2}\right)}{1.2\ldots\left(\frac{n+1}{2}\right)}$ the last two new factors are equal, and therefore ${}_nC_r$ is greatest in value when $r = \dfrac{n+1}{2}$ or $\dfrac{n-1}{2}$.

(ii.) Let n be *even*. Then when $r = \dfrac{n}{2}$,

$${}_nC_r = \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}}, \text{ and } {}_nC_r \text{ is greatest in value.}$$

7. The following results are of great importance in working out examples.

(i.) Suppose there are 2 sets (A, B) containing respectively m, n things; then the number of ways in which a set, which shall contain p things from A and q things from B, can be selected $= {}_mC_p \times {}_nC_q$. For each selection from A can be combined with each selection from B.

(ii.) To find the number of permutations under the same conditions, we must suppose first a selection of $(p+q)$ things to be made, and then these $(p+q)$ things arranged amongst themselves in the $\underline{|p+q}$ possible ways. We thus get ${}_mC_p \times {}_nC_q \times \underline{|p+q}$.

These two results may be extended to any number of sets A, B, C . . .

(iii.) To find the number of permutations or combinations under certain restrictions as in the following example :—In how many ways can a party of 4 ladies and 4 gentlemen be chosen from 10 ladies and 8 gentlemen, so as always to include a particular lady and at the same time to exclude a particular gentleman ?

In this case since we have one lady fixed we must choose 3 more from the remaining 9; this can be done in ${}_9C_3$ different ways. And since 1 gentleman is entirely excluded, we have to choose 4 from the remaining 7; this can be done in ${}_7C_4$ ways. Therefore the number of different ways $= {}_9C_3 \times {}_7C_4$

$$= \frac{9.8.7}{1.2.3} \times \frac{7.6.5.4}{1.2.3.4} = 2940.$$

8. If an event must happen in one of n different and equally likely ways, then the *chance* or *probability* of the event happening in any particular one of the n ways is $\frac{1}{n}$, if we represent certainty by 1.

For let $x =$ the required chance; then since the event must happen, $\therefore$ the sum of all the n separate equal probabilities must be certainty,

$$\therefore nx = 1, \text{ and } \therefore x = \frac{1}{n}.$$

e.g. If a sovereign is known to be in one of 10 boxes, the chance of opening at random the right box and finding the sovereign is $\frac{1}{10}$. And the *expectation* or money value of the chance would be $\frac{1}{10}$ of £1 = 2s.

9. Similarly the chance of an event happening in any one of m particular ways out of the n different ways is $\frac{m}{n}$.

e.g. If it is known that 3 out of the 10 boxes each contain a sovereign, the chance of finding a sovereign at the first trial is $\frac{3}{10}$; and the *expectation* in this case $=\frac{3}{10}$ of £1 = 6s.

10. By the *odds for* or *against* an event is meant the ratio of the chance of its happening to the chance of its failing, or *vice versa.*

Thus if an event may happen in a ways and fail in b, the chances of its happening and failing are $\frac{a}{a+b}$ and $\frac{b}{a+b}$ respectively; and the *odds* for and against are $a:b$ and $b:a$ respectively.

Again, if there are are three events A, B, C which can happen in a, b, c ways respectively, then the chance of A's happening is $\frac{a}{a+b+c}$; of B's is $\frac{b}{a+b+c}$, and of C's $\frac{c}{a+b+c}$, and similarly for any number of events.

e.g. A bag contains 3 white, 4 red, and 5 black balls. One is draw at random. The chance that it is white $=\frac{3}{3+4+5}=\frac{3}{12}=\frac{1}{4}$; that it is red $=\frac{4}{12}=\frac{1}{3}$; and that it is black $=\frac{5}{12}$. And the sum of these separate chances is $\frac{1}{4}+\frac{1}{3}+\frac{5}{12}=1$, as it should be, for it is certain that the ball drawn must be either white, red, or black.

11. *Compound Chances.*—Let the chance of the happening of an event (A) depend upon two other events (B) and (C), of which B can happen in a_1 ways and fail in b_1, and C can happen in a_2 ways and fail in b_2; so that their separate simple chances of happening are $\frac{a_1}{a_1+b_1}$ and $\frac{a_2}{a_2+b_2}$.

Then the whole number of favourable cases will be found by combining each favourable case of B with each favourable case of C, and is therefore equal to a_1a_2; also the total number of cases possible $=(a_1+b_1)(a_2+b_2)$ in the same way.

Therefore by first principles *chance of A happening* $=\frac{a_1a_2}{(a_1+b_1)(a_2+b_2)}=\frac{a_1}{a_1+b_1}\times\frac{a_2}{a_2+b_2}=$ *product of the simple chances of the separate events.*

Similarly the chance of both B and C failing $=\frac{b_1b_2}{(a_1+b_1)(a_2+b_2)}$; of B failing and C happening $=\frac{b_1a_2}{(a_1+b_1)(a_2+b_2)}$; and of B happening and C failing $=\frac{a_1b_2}{(a_1+b_1)(a_2+b_2)}$. The sum of these separate chances

$$=\frac{a_1a_2+b_1b_2+b_1a_2+a_1b_2}{(a_1+b_1)(a_2+b_2)}=1.$$

e.g. The chance that an ace will be thrown twice in succession by a single die $=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$.

Similarly, if A depends upon more than two events, the *compound* chance A will be the *product* of the separate *simple* chances upon which it depends.

12. We will consider next the case of successive trials where the number of favourable (a) and unfavourable (b) cases for each event remains the same. By § 11 the chance that in n trials the event should happen *each* time $=\frac{a^n}{(a+b)^n}$, and that it should fail *each* time $=\frac{b^n}{(a+b)^n}$. That it should happen only one *specified* time, and therefore fail the other $n-1$ times $=\frac{ab^{n-1}}{(a+b)^n}$; but if it may happen any one time, the possible ways are increased n times, and the chance would be $\frac{nab^{n-1}}{(a+b)^n}$. That it should happen only r *specified* times, and fail in the remaining $n-r$ trials $=\frac{a^rb^{n-r}}{(a+b)^n}$, but if it may happen *any* r times, the possible ways are increased $\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}$ times, this fraction being the

number of combinations of n things r at a time, that is, the number of ways in which the r times may occur.

e.g. The chance that in 5 throws with a single die the second and third *only* should be aces is $\frac{1^2 5^3}{(1+5)^5}=\frac{125}{7776}$.

And the chance that *any* 2 throws out of 5, but 2 *only*, should be aces, is $\frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{1^2 \cdot 5^3}{(1+5)^5}=\frac{1250}{7776}=\frac{625}{3888}$.

13. *Skill.* By a man's skill in a game is meant the chance of his winning any given game; for instance, if A's skill is to B's skill as 2 : 3, then on an average A wins 2 games out of 5, or his chance of winning any particular game is $\frac{2}{5}$.

EXAMPLES. VIII.

1. How many words of 4 consonants and 1 vowel can be made out of 12 consonants and 5 vowels?

2. How many different permutations can be made of the letters of the word *essences*, and how many of these will begin with n and end with s?

3. If ${}_nC_r$ is the number of combinations of n things taken r together, show that $n \times {}_{n-1}C_{r-1}=r \times {}_nC_r$, assuming that ${}_nC_r=\frac{\lfloor n}{\lfloor r \lfloor n-r}$.

4. Find the number of different permutations that can be made of the letters in $a^2b^3c^4$ written at full length.

5. In how many selections of 5 things out of 10 will two particular things a and b occur?

6. In how many different ways may the letters of the word *observatory* be written?

7. Find the number of different triangles which may be formed by joining the angular points of a polygon of m sides inscribed in a circle,—*e.g.* a hexagon.

8. Given that ${}_nP_r : {}_nP_{r-1} :: 10 : 1$, and ${}_nC_r : {}_nC_{r-1} :: 5 : 3$, find n and r.

9. If the faces of one of two dice were marked from 1 to 6, and of the other from 5 to 10, how many different throws could be made?

10. Find the value of n when ${}_nC_3 = 12\,{}_{\frac{n}{2}}C_3$.

11. In how many ways can the crew of an eight-oar be selected from 12 men, of whom 10 can row and cannot steer, one can steer and cannot row, and one can both row and steer?

12. A committee of 7 is to be chosen from 13 candidates, of whom 6 are Liberals and 7 Conservatives. In how many ways can the selection be made so as to give a Liberal majority?

13. In how many ways may the 5 letters a, b, c, d, e be arranged so that no one of them is removed more than one place at most from its alphabetical order?

14. On a railway there are 15 stations; find how many tickets are required in order to travel from any one station to any other.

15. Find all the permutations which can be made out of the letters of the word *Baccalaureus* taken all together.

16. Out of a company of soldiers, pickets of 3 were to be selected, and it was found that the number of different pickets was 1027 times the number of men in the company. What was that number?

17. The number of combinations of 10 things r together, when two specified things occur in each combination, is one-fifteenth of the whole number of combination of 10 things r together. Find r.

18. In how many ways may 3 scholarships be awarded amongst 36 candidates—(i.) neglecting the order in which any 3 selected are placed; (ii.) taking the order into account?

19. A party of 5 ladies and 3 gentlemen is to be selected from 10 ladies and 8 gentlemen; in how many ways can this be done? In how many of these ways will a particular lady be included and a particular gentleman excluded?

20. In how many ways can a necklace be strung with 3 red beads, 4 blue, 5 yellow, and 10 white?

21. Find the value of n when the number of permutations of n things 4 together is 15 times the number of combinations of $n+1$ things 3 together.

22. One family consists of 4 sons and 5 daughters, and another family of 7 sons and 6 daughters. In how many ways could a marriage be arranged between the two families, provided only that the youngest son or daughter of one family does not marry the eldest daughter or son of the other?

23. A bag contains n sovereigns and n shillings. Prove that the number of different ways in which they can be drawn out in succession, one at a time, is $\dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{\underline{|n}} \times 2^n$.

24. There are 5 flags of one kind, 7 of another, and 8 of a third kind. How many different signals may be made, each consisting of one flag of each kind, the order of the three flags in each signal being taken into account?

25. In how many ways may 8 be thrown with a pair of ordinary dice?

26. What are the odds against a man throwing an ace twice in succession with a single die?

27. Suppose two bags, each to contain a white counters and b black. If a man draw a counter out of each bag, what is the chance that they should be a white and a black?

28. Three bags contain each 1 red and 6 blue balls; a ball is drawn at random from each; find the chance—(i.) that all three should be red; and (ii.) that all three should be blue.

29. A board is marked with white and black one-inch squares, like a chess-board; a circular counter, diameter half-an-inch, is dropped upon it. Find the chance that it should not touch a black square.

30. A bag contains 5 tickets worth 1s. each, 7 worth 2s. 6d., and 12 worth 5s. If a man is allowed to draw 2, what is the value of his expectation?

31. If the chance that you can solve a certain problem is $\frac{1}{5}$, and the chance that two other particular persons cannot solve it between them is $\frac{3}{8}$, what is the chance that the problem will be solved among the three?

32. Three cards are taken at random from an ordinary pack; find the chance that they will consist of a knave, a queen, and a king.

33. A bag contains three red balls and two white ones; what is the

probability of taking out one white ball at least in two trials, the ball which is taken out first being put back again before the second trial?

34. If a bag contains 8 white balls and 5 black, and 7 be taken out at random, what is the chance that they will consist of 4 white and 3 black?

35. A and B play at a game which cannot be drawn, and on an average A wins three games out of every five. What is the chance that A should win three games at least out of the first five?

36. A ball, 1 inch in diameter, is thrown against a wire netting, the apertures of which are squares of 3 inches a side; find the chance that it goes through without hitting a wire. The thickness of the wire may be neglected.

37. What is the chance of drawing the four knaves in succession from a pack of cards?

What is the chance of drawing an ace, king, and queen in succession?

38. What is the chance of drawing out all one suit from a pack in succession?

39. What are the chances of throwing 7, 8, and 9 respectively in one throw with two dice?

40. A plays 7 games with B, and A's skill is to B's skill as 2 : 3. Find the chance that A wins 2 games only out of the 7.

41. Find the chance of throwing an ace and a deuce in two successive throws with the same die.

42. A bag contains 3 white, 4 black, and 7 red balls. Find the chance of drawing a white, a black, and a red in order in 3 successive trials.

43. Find the chance of throwing 3 aces exactly in 5 throws with a single die.

44. Find the chance of throwing 7 once at least in two trials with a pair of ordinary dice.

45. Find the odds against a man throwing 5 once and 7 once with 2 dice in three trials.

46. Compare the chance of throwing an ace with 2 dice, with the chance of throwing 5 points.

47. The chances that each of two men will die in a certain time are $\frac{5}{6}$ and $\frac{4}{5}$ respectively. Find the two probabilities that they will not both be dead, and that they will not both be alive at the end of that time.

48. Given that the probability of a certain event happening is a, while that of a certain other event happening is b, find the probability of both events happening—(i.) when the two events are independent; (ii.) when the second event cannot happen unless the first does, and its probability in that case is b.

49. Find the chance of throwing fives with two dice in three trials.

50. From a bag containing 2 guineas, 3 sovereigns, and 5 shillings, a person is allowed to draw 3 coins indiscriminately. Find the value of his expectation.

51. A and B play at chess and A wins on an average 2 games out of 3. Find the chance of A winning 4 games out of the first 6 which are not drawn.

52. Four flies come into a room in which there are 4 lumps of sugar of different degrees of attraction proportional to the numbers 8, 9, 10, 12. Find the chance that they will all select different lumps.

53. In a certain book-shelf 10 volumes consisting of 4 works of 1, 2, 3, and 4 volumes respectively, are placed at random. Find the chance (i.) that all the volumes of each work may be together; and (ii.) that they may be in order.

IX.—SCALES OF NOTATION AND PROPERTIES OF NUMBERS.

1. *Definition.*—The *Radix* of a *Scale of Notation* is the positive integer which determines the local value of the digits of any number expressed in that scale; *e.g.* in the ordinary scale (*radix* 10) 4375 is an abbreviation for $(4\times1000)+(3\times100)+(7\times10)+5$, *i.e.* $4\,.\,10^3+3\,.\,10^2+7\,.\,10+5$; in the duodenary scale (*radix* 12) 895 stands for $8\,.\,12^2+9\,.\,12+5$, and in the scale, *radix* r, would stand for $8\,.\,r^2+9\,.\,r+5$.

2. Any integer (N) may be expressed in a scale with any given integer (r) for its radix.

For if we divided N by the highest power of r contained in it, and the remainder by the next highest, and so on, we should at last get a remainder less than r, and the successive quotients and this last remainder would be the successive digits in the new scale reckoned from left to right. This method would, however, be very laborious, and a simpler method is given in the next article.

3. (i.) To change a given integer (N) from any one scale to any other.

Let the radix of the new scale be r.

Then $N=p_n r^n + p_{n-1} r^{n-1} + \ldots + p_1 r + p_0$, where $p_n p_{n-1} \ldots$ are at present unknown positive integers less than r, or any one, except p_n, may be zero.

Divide both sides by r: thus

$$Q_0 + \frac{R_0}{r} = p_n r^{n-1} + p_{n-1} r^{n-2} + \ldots + p_1 + \frac{p_0}{r};$$

Therefore $p_0 = R_0$; and the other digits $p_1 \ldots p_n$ are found in the same way by dividing the quotients $Q_0\ Q_1\ Q_2 \ldots$ by r, and taking the successive remainders $R_1\ R_2 \ldots$

(ii.) To express any fraction (F) in terms of radix fractions in any required scale.

Let $F = \frac{n_1}{r} + \frac{n_2}{r^2} + \frac{n_3}{r^3} + \ldots$ where $n_1\ n_2 \ldots$ are positive integers less than r, or any one may be zero.

Multiply both sides by r: thus $I_1 + F_1 = n_1 + \frac{n_2}{r} + \frac{n_3}{r^2} + \ldots$

Where I_1 is an *integer* and F_1 a *proper fraction*, and $\frac{n_2}{r} + \frac{n_3}{r^2} + \ldots$ cannot

be $> \frac{r-1}{r} + \frac{r-1}{r^2} + \ldots$ to ∞ *i.e.* $> \dfrac{\frac{r-1}{r}}{1-\frac{1}{r}}$ *i.e.* > 1.

Therefore $n_1 = I_1$, and the other digits $n_2, n_3 \ldots$ are found in the same way by taking the integral parts $I_2, I_3 \ldots$ of the products of the fractions $F_1, F_2 \ldots$ by r.

It will generally be best to express radix fractions in a decimal form. *See* § 4.

4. *Examples.*—(i.) Express 923·125 in the *septenary* scale.

$$\begin{array}{rl} 7)\underline{923} & \\ 7)\underline{131},6 & \\ 7)\underline{18},5 & \\ 2,4 & \end{array} \qquad \begin{array}{r} \cdot125 \\ \underline{7} \\ 0\cdot875 \\ \underline{7} \\ 6\cdot125 \end{array}$$

$\therefore$ 923·125 in the *denary* scale $=2456\cdot\dot{0}\dot{6}$ in the *septenary.*

(ii.) Convert 78t53 from the *undenary* scale to the *duodenary.*

$$\begin{array}{rl} 12)\underline{78t53} & \\ 12)\underline{7187},7 & \\ 12)\underline{662},5 & \\ 12)\underline{60},2 & \\ 5,6 & \end{array}$$

$\therefore$ 78t53 in the *undenary*
$=56257$ in the *duodenary.*

[t, e, d are employed as abbreviations for 10, 11, and 12 when the radix of the scale is >10.]

(iii.) In what scale is 673 equal to the *octonary* number 1446?

$$6r^2+7r+3=1\times8^3+4\times8^2+4\times8+6=806,$$

$$\therefore 6r^2+7r-803=0, \quad \therefore (6r+73)(r-11)=0,$$

$$\therefore r=11. \quad \text{That is, 673 is in the } \textit{undenary} \text{ scale.}$$

(iv.) Multiply $t1t3$ by $7e8$ in the *duodenary* scale, and verify the result by division.

$$\begin{array}{r} t1t3 \\ \underline{7e8} \\ 692t0 \\ 93849 \\ \underline{5e0e9} \\ \underline{68e5470} \end{array} \qquad \begin{array}{l} 7e8)\,68e5470\,(t1t3 \\ \underline{6788} \\ 1294 \\ \underline{7e8} \\ 6987 \\ \underline{6788} \\ 1ee0 \\ \underline{1ee0} \end{array}$$

(v.) Find the least number which, when expressed in the scale of 3, has zero for its 3 right-hand digits; when in the scale of 4, has 2 zero digits on the right; when in the scale of 5, has one zero digit on the right.

The number must be given by the simplest integral values of x, y, z which satisfy $x\,.\,3^3=y\,.\,4^2=z\,.\,5$.

$\therefore x=4^2\,.\,5=80$, and $\therefore$ the number is 2160 in the ordinary scale.

(vi.) Find a number of 6 digits having 7 in the unit's place, such that if it be multiplied by 3 the product is the same as the number obtained by transferring the first digit on the left to the right of the number.

If we multiply any number ending in 7 by 3, the product must end in 1. Therefore 1 must be the first digit on the left of N.

$\therefore$ we may assume $N=1\,.\,10^5+a\,.\,10^4+b\,.\,10^3+c\,.\,10^2+d\,.\,10+7,\quad (\alpha.)$

$\therefore 3N=a\,.\,10^5+b\,.\,10^4+c\,.\,10^3+d\,.\,10^2+7\,.\,10+1,\quad (\beta.)$

also from (α) $10N=10^6+a\,.\,10^5+b\,.\,10^4+c\,.\,10^3+d\,.\,10^2+7\,.\,10,\quad (\gamma.)$

Subtract (β) from (γ), thus $7N=10^6-1=999{,}999$,

$$\therefore N=\frac{999999}{7}=142857.\quad \text{Ans.}$$

5. The *greatest* and *least* numbers of n digits in the scale radix r, are evidently $(r-1)(r^{n-1}+r^{n-2}+\ldots+1)$, each digit being $r-1$, *i.e.* r^n-1; and $1\,.\,r^{n-1}+0\,.\,r^{n-2}+\ldots+0$, the first digit being 1 and the rest 0, *i.e.* r^{n-1}.

e.g. The greatest and least 4-figure numbers in the senary scale are 6^4-1 and 6^3, *i.e.* 1295 and 216, when expressed in the ordinary scale.

6. To weigh any weight (*e.g.* 341 lbs.) by means of a series of 1 lb., 3 lbs., 3^2 lbs. . . . any one of which may be placed in *either* scale-pan.

$$\begin{array}{rl}
3)\underline{341} & \\
3)\underline{114}, & -1 \\
3)\underline{38}, & 0 \\
3)\underline{13}, & -1 \\
3)\underline{4}, & 1 \\
1, & 1
\end{array}$$

Therefore we must put 3^5, 3^4, 3^3 in one scale-pan, and 3^2, 1 in the other.

7. (i.) Any number in the scale of r divided by $r-1$ will leave the same remainder as the sum of its digits when divided by $r-1$.

For $$\frac{p_nr^n+\ldots+p_1r+p_0}{r-1}=\frac{p_n(r^n-1)+\ldots+p_1(r-1)+p_n+\ldots+p_1+p_0}{r-1}$$

$$=\text{Integer}+\frac{p_n+\ldots+p_1+p_0}{r-1}.$$ [See § 10, p. 2.]

Therefore any number in the scale of r is divisible by $r-1$ if the sum of its digits is divisible by $r-1$.

(ii.) Any number in the scale of r divided by $r+1$ will leave the same remainder as the difference between the sum of its odd digits and the sum of its even digits when divided by $r+1$.

For $\dfrac{p_nr^n+\ldots+p_1r+p_0}{r+1}$

$$=\frac{p_n\{r^n-(-1)^n\}+\ldots+p_1(r+1)+(-1)^np_n+\ldots-p_1+p_0}{r+1}$$

$$=\text{Integer}+\frac{(-1)^np_n+\ldots-p_1+p_0}{r+1}.\quad \text{[See § 10, p. 2.]}$$

Therefore any number in the scale of r is divisible by $r+1$ if the difference between the sum of its odd digits and the sum of its even digits is either zero or divisible by $r+1$.

8. *Properties of numbers in the ordinary scale.*

(i.) Any number when divided by 3, leaves the same remainder as the sum of its digits when divided by 3.

For $\dfrac{p_n10^n+\ldots+p_210^2+p_110+p_0}{3}$

$$=\frac{p_n(10^n-1)+\ldots+p_299+p_19+p_n+\ldots+p_1+p_0}{3}$$

$$=p_n\frac{10^n-1}{3}+\ldots+p_233+p_13+\frac{p_n+\ldots+p_1+p_0}{3}$$

$$=\text{Integer}+\frac{p_n+\ldots+p_1+p_0}{3}.$$

Therefore a number is divisible by 3 if the sum of its digits is divisible by 3.

(ii.) A number is divisible by 4 if its last two digits form a number which is divisible by 4.

For any number N may be put into the form $A\,.\,10^2+p_110+p_0$.

$$\therefore \frac{N}{4}=A\,.\,25+\frac{p_110+p_0}{4}.$$

(iii.) A number is divisible by 6 if it is *even*, and if also the sum of its digits is divisible by 3. [See (i.).]

(iv.) A number is divisible by 8 if its last 3 digits form a number divisible by 8.

The proof is similar to (ii.)

(v.) Any number, when divided by 9, leaves the same remainder as the sum of its digits when divided by 9. Therefore a number is divisible by 9 if the sum of its digits is divisible by 9.

[See § 7 (i.).]

(vi.) A number is divisible by 11 if the difference between the sum of its odd digits and the sum of its even digits is 0, or divisible by 11.

(vii.) A number is divisible by 12 if both (i.) and (ii.) are satisfied.

(viii.) To test the accuracy of multiplication by casting out the nines.

Any two numbers A and B may be put into the forms $9x+a$ and $9y+b$ where all the letters represent positive integers.

$\therefore\ A\times B=81xy+(bx+ay)9+ab=M(9)+ab$, where $M(9)$ is used as an abbreviation for *multiple of* 9, $\therefore$ the product of A and B divided by 9 leaves same remainder as ab divided by 9.

e.g. To *test* the truth of $257\times 606=155742$.

$$\frac{2+5+7}{9}=1+\frac{5}{9},\ \therefore\ a=5\text{; and }\frac{6+0+6}{9}=1+\frac{3}{9},\ \therefore\ b=3,$$

$$\therefore\ \frac{ab}{9}=1+\frac{6}{9}.\quad\text{Also }\frac{1+5+5+7+4+2}{9}=\frac{24}{9}=2+\frac{6}{9}.$$

Therefore this test is satisfied, but there may be errors which it fails to detect.

(ix.) Every square number is of one of the forms $5n$ or $5n\pm 1$.

Any number is of one of the forms $5m$, $5m\pm 1$ or $5m\pm 2$. Squaring these we get

$25m^2$, $25m^2 \pm 10m + 1$, and $25m^2 \pm 20m + 4$, *i.e.* $25m^2 \pm 20m + 5 - 1$, which are of the forms

$$5n,\ 5n+1, \text{ and } 5n-1.$$

(x.) The product of any number (r) of consecutive integers is divisible by $\lfloor r$.

For $\dfrac{n(n-1)\dots(n-r+1)}{\lfloor r} = {}_nC_r$ [see p. 60, § 3], *i.e.* the number of combinations of n things r at a time, which from its meaning is necessarily a *whole* number.

(xi.) To prove the rule for converting a Recurring Decimal (D) into an equivalent Vulgar Fraction.

Let $D = \cdot PQQQ \dots$ where P is the part that does not recur and consists of p digits, and Q is the recurring part consisting of q digits.

Thus

$$D = \frac{P}{10^p} + \left(\frac{Q}{10^{p+q}} + \frac{Q}{10^{p+2q}} + \frac{Q}{10^{p+3q}} + \dots \text{to } \infty\right)$$

$$= \frac{P}{10^p} + \frac{Q}{10^{p+q}\left(1 - \frac{1}{10^q}\right)}. \quad \text{[See p. 52, § 3.]}$$

$$= \frac{P}{10^p} + \frac{Q}{10^p(10^q - 1)}$$

$$= \frac{(P.10^q + Q) - P}{(10^q - 1)10^p}.$$

Therefore D = a fraction whose numerator is got by subtracting the non-recurring part from the whole number, and whose denominator consists of as many *nines* as there are figures in the recurring part followed by as many *ciphers* as there are figures in the non-recurring part.

(xii.) A *prime* is a number which is divisible by no integer but 1.

e.g. 1, 3, 5, 7, 11, 13, 17 . . . are *primes*.

(xiii.) Two numbers are said to be *prime* to one another when they have no common divisor but 1.

e.g. 7 and 12 ; 36 and 55.

(xiv.) The number of primes is infinite.

For if we take all the primes already found, multiply them together and add 1 to their product, it is evident that the result is divisible by none of them, and is therefore a new prime.

[In the following §§ 9-14 only *positive* quantities are considered.]

9. The Arithmetic Mean is greater and the Harmonic Mean is less than the Geometric Mean between the same two quantities.

For $(\sqrt{a}-\sqrt{b})^2 = a - 2\sqrt{ab} + b$ is always>0, every square being necessarily a *positive* quantity, $\therefore a+b > 2\sqrt{ab}$, $\therefore \frac{a+b}{2} > \sqrt{ab}$ *i.e.* A.M.>G.M.

Again, by § 7, p. 53, (H.M.) × (A.M.) = (G.M.)^2, $\therefore$ H.M. is <G.M.

10. If there are any number of unequal fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2} \dots$ then $\frac{m_1a_1+m_2a_2+\dots}{m_1b_1+m_2b_2+\dots}$ lies in value between the *greatest* and the *least* of the given fractions.

For let $\frac{a_r}{b_r}, \frac{a_s}{b_s}$ be the greatest and least fractions respectively,

$$\text{then } \frac{a_1}{b_1} \text{ is} < \frac{a_r}{b_r}, \therefore \frac{m_1a_1}{m_1b_1} < \frac{a_r}{b_r}, \quad \therefore m_1a_1 < \frac{a_r}{b_r}m_1b_1.$$

$$\text{Similarly,} \quad m_2a_2 < \frac{a_r}{b_r}m_2b_2$$

$$\dots\dots\dots\dots$$

$$m_ra_r = \frac{a_r}{b_r}m_rb_r$$

$$\dots\dots\dots\dots$$

$$\therefore m_1a_1+m_2a_2+\dots+m_ra_r\dots < \frac{a_r}{b_r}(m_1b_1+m_2b_2+\dots m_rb_r+\dots)$$

$$\therefore \frac{m_1a_1+m_2a_2+\dots}{m_1b_1+m_2b_2+\dots} < \frac{a_r}{b_r}. \text{ Similarly we can prove it} > \frac{a_s}{b_s}.$$

If we put $m_1 = m_2 = \dots = 1$, we get that $\frac{a_1+a_2+\dots}{b_1+b_2+\dots}$ lies in value between the greatest and least of $\frac{a_1}{b_1}, \frac{a_2}{b_2} \dots$

11. (i.) If the *sum* of two quantities x and y is constant (C), their *product* xy is *greatest* when they are equal.

For, $xy = \frac{(x+y)^2-(x-y)^2}{4} = \frac{C^2-(x-y)^2}{4}$, which is the greatest when $x-y=0$, *i.e.* when $x=y$.

(ii.) *Conversely.* If the *product* of two quantities is constant (C), their sum is *least* when they are equal.

For then $(x+y)^2=(x-y)^2+4C$, $\therefore x+y$ is least when $x=y$.

12. The A.M. of any number of quantities $a_1, a_2, a_3 \ldots a_n$ is $>$ their G.M., *i.e.* $\frac{a_1+a_2+\ \ldots\ +a_n}{n}$ is $> (a_1a_2 \ldots a_n)^{\frac{1}{n}}$.

Let their sum be denoted by S, which is *constant:* then if any two, as a_r, a_s, are *unequal,* we can make their product greater, without altering their sum by making them both equal to $\frac{a_r+a_s}{2}$ [see § 11 (i.)]; and by proceeding in this way we can make the quantities as nearly equal as we please. Thus the product $a_1a_2 \ldots a_n$ is greatest when they are all equal, *i.e.* when each $=\frac{S}{n}$, and then the product $=\left(\frac{S}{n}\right)^n$.

Therefore $\left(\frac{S}{n}\right)^n$ is $> a_1a_2 \ldots a_n$,

$$\therefore \frac{a_1+a_2+\ \ldots\ +a_n}{n} \text{ is} > (a_1a_2 \ldots a_n)^{\frac{1}{n}}.$$

13. If a, b, c be such that any two are together $>$ the third,

$$\text{(i.)}\ 2(bc+ca+ab) \text{ is} > a^2+b^2+c^2,$$
$$\text{and (ii.)}\ (a+b+c)^2 \text{ is} > 2(a^2+b^2+c^2).$$

(i.) For $a+b>c$, $\therefore a>c-b$, $\therefore a^2>b^2-2bc+c^2$,

$\therefore 2bc$ is $>b^2+c^2-a^2$; similarly $2ca>c^2+a^2-b^2$ and $2ab>a^2+b^2-c^2$,

$$\therefore \text{by addition } 2(bc+ca+ab)>a^2+b^2+c^2.$$

(ii.) For $a(a+b+c)$ is $>a(a+a)$, *i.e.* $>2a^2$.

Similarly, $b(a+b+c)$ is $>2b^2$, and $c(a+b+c)$ is $>2c^2$.

And $\therefore$ by addition $(a+b+c)^2$ is $>2(a^2+b^2+c^2)$.

14. To prove that a^3+b^3 is $>a^2b+ab^2$.

Whether a be$>b$ or b be$>a$, (a^2-b^2) and $(a-b)$ have the same sign, therefore

$$(a^2-b^2)(a-b) \text{ is positive,}$$
$$\therefore a^3-a^2b-ab^2+b^3 \text{ is } >0,$$
$$\therefore a^3+b^3 \text{ is } >a^2b+ab^2.$$

Similarly it can be proved that

$a^{m+n}+b^{m+n}$ is $>a^mb^n+a^nb^m$ where m, n are any positive integers.

EXAMPLES. IX.

1. Given 222·22, in the scale whose radix is 5, reduce it to the denary scale.

2. Transform 25603 from the nonary to the septenary scale.

3. Multiply 1024 by 249 in the duodenary scale.

4. Find the square of 2341 in the quinary scale.

5. Divide 29*t*96580 by 2*tt*9 in the scale of 12.

6. Transform 23784 and 587 from the scale of 9 to that of 12.

7. Transform the ordinary number 26·5 to the quaternary scale.

8. Express 7631 in the scales of 2 and 20.

9. Find the square root of *eet*001 in the scale of 12.

10. How may 1319 lbs. be weighed by the series of weights 1 lb. 3 lbs., 9 lbs., 27 lbs., etc. ?

11. Transform 1·23 from the septenary to the denary scale.

12. In what scale will the double of the ordinary number 145 be expressed by the same digits ?

13. In what scale is the denary number 4954 expressed by 20305 ?

14. Express the circulating decimal $\cdot\dot{9}$ as a geometrical series, and hence find its value.

15. Find the number which is expressed by the same two digits, whether the scale be 7 or 9.

16. Find the square root of 25400544 in the senary scale.

17. How may 2304 in the quinary scale be expressed by terms of the series ± 1, ± 3, $\pm 3^2$, $\pm 3^3 \ldots$?

18. In what scale of notation will the ordinary number 16000 be expressed by 1003000?

19. There is a number consisting of three digits in G. P. The number is to the sum of its digits as 124 : 7, and if 594 be added to it, the digits will be inverted; what is it?

20. Prove that any number of 4 digits is divisible by 7, if the first and last digits be the same, and the digit in the hundreds' place be double that in the tens' place.

21. What is the least number by which 2205 must be multiplied so that the product may be a perfect cube?

22. Prove that $n^4 + 2n^3 - n^2 - 2n$ is divisible by 24 if n be any positive integer greater than 2.

23. Prove that the difference between any integer and its cube is a multiple of 6.

24. If $a^3 - b^3$ is divisible by 3, then $(a \pm k)^3 - (b \pm k)^3$ will also be divisible by 3 where k is any integer.

25. If n is an *odd* integer, $\dfrac{n(n^4 - 1)}{48}$ is an integer.

26. If n be any *odd square* number >1, $(n+3)(n+7)$ is divisible by 32.

27. No number which is a *perfect square* can, when divided by 3, leave 2 as a remainder.

28. If n be an *even* integer, then $n(n^2 + 20)$ is divisible by 24.

29. Every cube number is of the form $7n$ or $7n \pm 1$.

30. If the number expressed by the last n digits of a number be divisible by 2^n, the number itself is divisible by 2^n.

31. If n be any whole number >2, prove that $n(n^2 - 1)(n^2 - 4)$ is a multiple of 120.

32. If a, b, c are in H.P., prove $a^2 + c^2 > 2b^2$.

33. Prove that if n be any positive integer, $\frac{n(n+1)(2n+1)}{6}$ is an integer.

34. If there be n figures in the square root of a number, there cannot be more than $2n$ or less than $(2n-1)$ figures in the number itself.

35. If any four consecutive odd numbers be multiplied together, and 16 be added to the product, prove that the result is an exact square.

X.—BINOMIAL THEOREM.

1. *To find the product of any number of Binomial Factors of the form $(x+a)$, and thence to deduce the Binomial Theorem, that is, the expansion of $(x+a)^n$ in a series of descending powers of x, when n is a positive integer.*

(*a.*) By ordinary multiplication we get [see p. 2, § 9],

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc,$$

and $(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3$
$$+ (ab+ac+ad+bc+bd+cd)x^2 + (bcd+acd+abd+abc)x + abcd.$$

In each of these results we notice that the following laws hold :—

(i.) *Index* of highest power of x = number of Binomial Factors.

(ii.) *Co-efficient* of same = 1.

(iii.) *Indices* of *succeeding* terms decrease by 1.

(iv.) *Number* of terms = number of Factors + 1.

(v.) *Co-efficient* of 2d Term = Sum of 2d Terms of Binomial Factors ;
 „ 3d Term = Sum of Products of 2d Terms 2 at a time ;
 „ 4th Term = „ „ 3 „
 etc. etc. etc. . . .

(vi.) *Last* Term = Product of 2d Terms of Factors.

Now *assume* that these laws hold for $(n-1)$ factors where n is any positive integer; that is, that

$$(x+a)(x+b)\ldots(x+k)=x^{n-1}+p_1x^{n-2}+p_2x^{n-3}+\ \ldots+p_{n-1}$$

where $p_1=a+b+c+\ldots+k$.

$p_2=ab+ac+bc+\ldots=$ sum of products 2 at a time;

$p_3=abc+\ldots\quad=\quad$ „ $\quad$ 3 $\quad$ „

$\ldots\ \ldots\ldots$

$p_{n-1}=abc\ldots k\quad=$ product of all the second terms.

Multiply both sides by a new n^{th} binomial factor $x+l$.

Thus $(x+a)(x+b)(x+c)\ldots(x+k)(x+l)$

$$=x^n+(p_1+l)x^{n-1}+(p_2+lp_1)x^{n-2}+(p_3+lp_2)x^{n-3}+\ldots+p_{n-1}l.$$

Laws (i.), (ii.), (iii.), (iv.) evidently are still true; and laws (v.) and (vi.) are true,

$\because p_1+l=a+b+\ldots+k+l\quad=$ sum of 2d terms;

$p_2+lp_1=(ab+ac+\ldots)+(a+b+\ldots+k)l=$ sum of products 2 at a time;

$p_3+lp_2=(abc+\ldots)+(ab+ac+\ldots)l\quad=\quad$ „ $\quad$ 3 $\quad$ „

$\ldots\ \ldots\ldots$

and $p_{n-1}l=abc\ldots k\,.\,l\quad=$ product of all the 2d terms.

Therefore if the laws hold for $n-1$ factors they hold for n, but we have proved them true by ordinary multiplication for 4 factors; therefore they are true for 5, 6 ... and therefore universally.

(β.) Now suppose $b=c=\ldots=l=a$.

Then $(x+a)(x+b)\ldots(x+l)=(x+a)^n$.

And $p_1+l=na: p_2+lp_1=\dfrac{n(n-1)}{\underline{|2}}a^2$ for each product $=a^2$, and their number is the same as the number of combinations of n things 2 at a time. Similarly, $p_3+lp_2=\dfrac{n(n-1)(n-2)}{\underline{|3}}a^3$, etc.; and $p_{n-1}l=a^n$.

Therefore

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{\underline{|2}}a^2x^{n-2}+\frac{n(n-1)(n-2)}{\underline{|3}}a^3x^{n-3}+\ldots+a^n.$$

2. By considering the first few terms of this series, we see that the r^{th} term *from the beginning* is

$$\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}a^{r-1}x^{n-r+1}=\frac{\lfloor n}{\lfloor r-1\lfloor n-r+1}a^{r-1}x^{n-r+1}.$$

Again, $\because$ the series consists of $n+1$ terms, therefore the r^{th} term *from the end* is the $(n+2-r)^{th}$ term *from the beginning*, and therefore is

$$\frac{n(n-1)\ldots(n-\overline{n+2-r}+2)}{\lfloor n+2-r-1}a^{\overline{n+2-r}-1}x^{n-\overline{n+2-r}+1}$$

$$=\frac{n(n-1)\ldots r}{\lfloor n-r+1}a^{n-r+1}x^{r-1}$$

$$=\frac{\lfloor n}{\lfloor n-r+1\lfloor r-1}a^{n-r+1}x^{r-1}.$$

Therefore the numerical co-efficients of the r^{th} terms from the beginning and from the end are the same; that is, after the middle of the expansion the co-efficients are repeated in inverse order, and therefore may be written down without calculation.

Again, by § 6, p. 61, we see that the greatest numerical co-efficient is that of the two middle terms, or of the middle term, according as n is *odd* or *even*.

3. In the above proof x and a are unrestricted in value, we may therefore write $-a$ for a; thus we get

$$(x-a)^n=x^n-nax^{n-1}+\frac{n(n-1)}{\lfloor 2}a^2x^{n-2}-\ldots$$

Again, if we write 1 for x and x for a, we get

$$(1\pm x)^n=1\pm nx+\frac{n(n-1)}{\lfloor 2}x^2\pm\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\ldots$$

4. Next, let $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$

so that $c_0=1$, $c_1=n$, $\ldots$ $c_r=\dfrac{n(n-1)\ \ldots\ (n-r+1)}{\lfloor r}$.

(*a.*) Put $x=1$, then $2^n=c_0+c_1+c_2+\ldots+c_n$.

[This gives us that the total number of different ways of taking n things 1, 2, 3 . . . n at a time$=2^n-1$.]

(*b.*) Put $x=-1$, then

$$0=c_0-c_1+c_2-c_3+\ldots$$

$$\therefore c_0+c_2+c_4+\ldots=c_1+c_2+c_3+\ldots\ldots=2^{n-1} \text{ by } (a.)$$

(*c.*) Put $\frac{1}{x}$ for x, then

$$\left(1+\frac{1}{x}\right)^n=c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots$$

$$\therefore (1+x)^n\left(\frac{1+x}{x}\right)^n=(c_0+c_1x+\ldots)\left(c_0+\frac{c_1}{x}+\ldots\right)$$

$$\therefore (1+x)^{2n}=x^n\left\{\ldots+\frac{a}{x^2}+\frac{b}{x}+(c_0{}^2+c_1{}^2+\ldots)+cx+dx^2+\ldots\right\}$$

$$\therefore c_0{}^2+c_1{}^2+c_2{}^2+\ldots=\text{co-efficient of } x^n \text{ in } (1+x)^{2n}$$

$$=\frac{2n(2n-1)\ldots(n+1)}{\underline{|n}}=\frac{\underline{|2n}}{(\underline{|n})^2}.$$

(*d.*) $\because (1+x)^n=c_0+c_1x+c_2x^2+\ \ldots+c_nx^n$

$$\therefore (1+x)^{n+1}=(1+x)^n\ (1+x)=(c_0+c_1x+\ldots)(1+x)$$

$$=c_0+(c_0+c_1)x+(c_1+c_2)x^2+\ldots+c_nx^{n+1}$$

That is, the co-efficient of the r^{th} term in the expansion of $(1+x)^{n+1}$ is got by adding together the co-efficients of the $(r-1)^{th}$ and r^{th} terms in the expansion of $(1+x)^n$.

If then we take successive integral positive values of n starting from 1, we get the following table of co-efficients.

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\end{array}$$

. . . .

and so on: each number in any row being the sum of the two numbers to its right and left in the row immediately above it.

5. To prove the Binomial Theorem for a *fractional* and for a *negative* exponent; that is, that

$$(1+x)^m=1+mx+\frac{m(m-1)}{\underline{|2}}x^2+\ldots$$

whatever m may be, the co-efficients following the same law of formation as when m is a *positive integer*.

Let $f(m)$ stand for $1+mx+\frac{m(m-1)}{\underline{|2}}x^2+\ldots$ whatever m may be.

Then, if m and n are *positive integers*, we have $f(m)=(1+x)^m$, $f(n)=(1+x)^n$, $f(m+n)=(1+x)^{m+n}$ by § 3; and by the Theory of Indices

$$(1+x)^m(1+x)^n=(1+x)^{m+n}=1+(m+n)x+\frac{(m+n)(m+n-1)}{\underline{|2}}x^2+\ldots$$

i.e. $f(m+n)=f(m)\,.\,f(n)$ when m and n are *positive integers.*

We *assume* next that the *form* of the product of the two series represented by $f(m)$ and $f(n)$ does not depend on the *values* of m and n, and will therefore always be the same and therefore equal to the series represented by $f(m+n)$.

Therefore $f(m+n)=f(m)\,.\,f(n)$ *universally.*

Similarly, $f(m)\,.\,f(n)\,.\,f(p)=f(m+n)\,.\,f(p)=f(m+n+p)$; and thus

$$f(m+n+p+q+\ldots)=f(m)\,.\,f(n)\,.\,f(p)\,.\,f(q)\ldots$$

Now put $m=n=p=\ldots=\frac{s}{r}$, s and r being *positive integers* and r being the number of equal fractions m, n, $p\ldots$

Thus $f(s)=\left\{f\left(\frac{s}{r}\right)\right\}^r \quad \therefore \{f(s)\}^{\frac{1}{r}}=f\left(\frac{s}{r}\right)$

But $\because$ s is a positive integer $\therefore \{f(s)\}^{\frac{1}{r}}=\{(1+x)^s\}^{\frac{1}{r}}=(1+x)^{\frac{s}{r}}$

$$\therefore (1+x)^{\frac{s}{r}}=f\left(\frac{s}{r}\right)=1+\frac{s}{r}x+\frac{\frac{s}{r}\left(\frac{s}{r}-1\right)}{\underline{|2}}x^2+\ldots$$

This proves the theorem for a positive fraction, and therefore it is true for *any positive quantity, integral or fractional.*

Next, let n be any positive quantity.

Then $f(-n)\,.\,f(n)=f(-n+n)=f(0)=1$.

$$\therefore f(-n)=\frac{1}{f(n)}=\frac{1}{(1+x)^n}=(1+x)^{-n}$$

$$\therefore (1+x)^{-n}=f(-n)=1-nx+\frac{-n(-n-1)}{\underline{|2}}x^2+\ldots$$

Which proves the theorem for *any negative quantity.*

The number of terms in this expansion is infinite, for in no case can we get a zero factor in the numerator of the co-efficient. The result is arithmetically intelligible only when the series is *convergent.* [See xii. §§ 1, 2.]

6. To expand $(a+x)^n$ when n is fractional or negative we may put it into the form $a^n\left(1+\frac{x}{a}\right)^n$ which by the above is equal to

$$a^n\left\{1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{\lfloor 2}\left(\frac{x}{a}\right)^2+\ldots\right\}.$$

7. The Binomial Theorem may be used to extract roots *approximately* by means of the formula $\sqrt[n]{a^n\pm b}=a\left(1\pm\frac{b}{a^n}\right)^{\frac{1}{n}}$.

e.g. $\sqrt{99}=(100-1)^{\frac{1}{2}}=(10^2-1)^{\frac{1}{2}}=10\left(1-\frac{1}{10^2}\right)^{\frac{1}{2}}$

$$=10\left\{1-\frac{1}{2}\cdot\frac{1}{10^2}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\cdot\frac{1}{10^4}-\ldots\right\}$$

$$=10(1-\cdot005-\cdot0000125)=10\times\cdot9949875=9\cdot949875.$$

8. *Examples.*

(i.) Find the middle term in the expansion of $(2a+3x^2)^8$.

There are 9 terms $\therefore$ the middle term is the fifth, which is

$$\frac{8\,.\,7\,.\,6\,.\,5}{4\,.\,3\,.\,2}(2a)^4(3x^2)^4=7\,.\,2\,.\,5\,.\,16\,.\,81\,.\,a^4x^8=90720a^4x^8.$$

(ii.) Expand $(1-x)^{-4}$ to 5 terms, and write down the co-efficient of x^m.

$$(1-x)^{-4}=1-4(-x)+\frac{-4(-4-1)}{\lfloor 2}(-x)^2+\text{etc.}\ldots$$

$$=1+4x+10x^2+20x^3+35x^4+\text{etc.}$$

And the co-efficient of x^m is $\frac{4\,.\,5\,.\,6\,\ldots\,(m+3)}{1\,.\,2\,.\,3\,\ldots\,m}=\frac{1}{6}\frac{\lfloor m+3}{\lfloor m}$.

(iii.) Prove that

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\,.\,\frac{x}{1+2x}+\frac{n(n+1)}{\lfloor 2}\left(\frac{x}{1+2x}\right)^2+\ldots$$

$$\left(\frac{1+2x}{1+x}\right)^n=\left(\frac{1+x}{1+2x}\right)^{-n}=\left(1-\frac{x}{1+2x}\right)^{-n}$$

$$=1+n.\frac{x}{1+2x}+\frac{-n(-n-1)}{\lfloor 2}\left(\frac{x}{1+2x}\right)^2+\ldots$$

$$=1+n.\frac{x}{1+2x}+\frac{n(n+1)}{\lfloor 2}\left(\frac{x}{1+2x}\right)^2+\ldots$$

(iv.) Find the value of $\dfrac{\sqrt{1+x}+(1-x)^{\frac{2}{3}}}{1+x+\sqrt{1+x}}$, when x is *very small.*

$$\text{Fraction } =\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{1+x+(1+x)^{\frac{1}{2}}}=\frac{(1+\frac{1}{2}x+\ldots)+(1-\frac{2}{3}x+\ldots)}{1+x+(1+\frac{1}{2}x+\ldots)}$$

$$=\frac{2-\dfrac{x}{6}}{2+\dfrac{3}{2}x}=\left(1-\frac{x}{12}\right)\left(1+\frac{3x}{4}\right)^{-1}=\left(1-\frac{x}{12}\right)\left(1-\frac{3x}{4}+\ldots\right)$$

$$=1-\frac{x}{12}-\frac{3x}{4}\ldots=1-\frac{5x}{6}.$$

Because x is very small we may neglect x^2 at every step in comparison with x.

EXAMPLES. X.

1. Find the eighth term in the expansion of $(1+x)^{11}$.

2. Expand $\dfrac{1}{\sqrt[3]{1-x}}$ to four terms.

3. Find the first negative term in the expansion of $\left(1+\dfrac{x^2}{3}\right)^{\frac{10}{3}}$, and the fourth term in $(2a-3x)^8$.

4. Write down the co-efficient of x^4 in the expansions of $(1-x)^{-3}$ and $(2-3x)^{\frac{5}{3}}$ by the binomial theorem.

5. Prove that the co-efficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is equal to $\dfrac{\underline{|2r}}{(\underline{|r})^2}$.

6. Write down the r^{th} term in the expansion of $(1-2x)^{-\frac{1}{2}}$.

7. Expand $(1-x^2)^{-\frac{3}{2}}$ to five terms.

8. Find the value of the greatest co-efficient in the expansion (i.) of $(a+b)^6$, and (ii.) of $(1+x)^9$.

9. If the co-efficient of the $(p+1)^{th}$ term of the expansion of $(1+x)^n$ is equal to that of the $(p+3)^{th}$, prove that $2p=n-2$.

10. Expand $(1+x)^{-\frac{1}{2}}$ in powers of x, writing down the co-efficient of x^r.

11. Expand $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^4$, and find the last three terms of $(x-y)^{15}$.

12. Find the first four terms of $\dfrac{1}{\sqrt{1-x^2}}$ and the r^{th} term of $(1-x)^{\frac{1}{2}}$.

13. Find the numerically greatest term in the expansion of $(a\pm x)^n$, where n is a positive integer.

14. Prove that $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}=1+\dfrac{x}{a+x}+\dfrac{3}{2}\dfrac{x^2}{(a+x)^2}+\dfrac{5}{2}\dfrac{x^3}{(a+x)^3}+\text{etc.}$

15. Expand $(a+b\sqrt{-1})^n$ to 4 terms.

16. Find the middle term in the expansion of $\left(x+\dfrac{1}{x}\right)^{2m}$ where m is a positive integer.

17. Find the fourth term of the expansion of $(a^{\frac{2}{3}}-b^{\frac{2}{3}})^{\frac{3}{2}}$.

18. Write down the co-efficients of x^r in the expansions of $(1-x)^{-2}$ and $(1-x)^{-\frac{1}{2}}$.

19. If (n) be a prime number, prove that every term in the expansion of $(1+x)^n$, except the first and last, is divisible by n.

20. Find the value of $\dfrac{a-(a^n-x^n)^{\frac{1}{n}}}{x}$ when $x=0$.

21. Expand $(1-3x)^{\frac{2}{3}}$ to 5 terms, and write down the general term in its simplest form.

22. If a_r denote the co-efficient of x^r in the expansion of $(1-x)^{2m-1}$ show that $a_{r-1}+a_{2m-r}=0$.

23. Express the sum of the co-efficients of a binomial whose index is a positive whole number in terms of a power of 2, and verify the property in the expansion of $(1+x)^6$.

24. Find the middle term in the expansion of $\left[5a-\dfrac{x}{5}\right]^{16}$.

25. In the expansion of $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$ in ascending powers of x, prove that the co-efficients of the $\overline{2r-1}|^{th}$ and the $\overline{2r}|^{th}$ terms are the same.

26. The co-efficient of the third term in the expansion of $(1-x)^{-n}$ is $\frac{2}{9}$; find (n) and the co-efficient of the fifth term.

27. Assuming the form of the expansion by the binomial theorem of $(1+x)^{n-1}$ when (n) is any positive whole number, show that the co-efficient of x^r in $(1+x)\,.\,(1+x)^{n-1}$ is the number of combinations of (n) things taken (r) together.

28. Prove that—

$$1+\frac{1}{6}+\frac{1\,.\,3}{1\,.\,2}\cdot\frac{1}{6^2}+\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\cdot\frac{1}{6^3}+\frac{1\,.\,3\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{1}{6^4}+\ldots=\sqrt{\frac{3}{2}}.$$

29. Show that the product of the sum of the *odd* terms and the sum of the *even* terms of the expanded binomial $(a+b)^m=\frac{1}{4}\{(a+b)^{2m}-(a-b)^{2m}\}$.

30. Prove that $\left(\frac{1+x}{1-x}\right)^n=1+n\,.\,\frac{2x}{1+x}+\frac{n(n+1)}{1\,.\,2}\left(\frac{2x}{1+x}\right)^2+\text{etc.}$

31. Find the value of

$$1-\frac{1}{2}\,.\,\frac{1}{2}+\frac{1\,.\,3}{2\,.\,4}\,\frac{1}{2^2}-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{1}{2^3}+\ldots\text{ to }\infty$$

32. If E denote the sum of the even terms and O the sum of the odd terms in the expansion of $(a+b)^n$, prove that $E^2\sim O^2=(a^2\sim b^2)^n$.

33. Prove that when x is very small

$$\frac{3(x+\frac{4}{9})^{\frac{1}{2}}(1-\frac{3}{4}x^2)^{\frac{1}{3}}}{2(1+\frac{9}{16}x)^2}=1-\frac{307}{256}x^2 \text{ approximately.}$$

34. Find the fourth root of $\frac{10{,}000}{9999}$ correct to 7 places of decimals by the Binomial Theorem.

35. Prove that $1-\frac{1}{2}+\frac{1\,.\,3}{2\,.\,4}\,\frac{1}{2^2}-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{2^3}+\ldots=\frac{1}{\sqrt{2}}.$

36. Find the co-efficient of x^{-15} in the expansion of $(1-x^{-3})^{-7}$.

37. Expand $x^m\left(1-\frac{1}{x}\right)^{-n}$ in a series of powers of x.

In the case in which m is a positive integer, write down the term which does not contain x, and prove that it will be a positive integer whenever n is a positive integer.

38. Find the co-efficient of x^{2n} in the expansion of $\frac{1-x}{(1+x)^2}$.

39. Find $\sqrt[3]{28}$, and $\sqrt[5]{3123}$ by the Binomial Theorem.

40. Prove that $\dfrac{\sqrt{1+x}+(1-x)^{\frac{2}{3}}}{1+x+\sqrt{1+x}}=\dfrac{1}{\sqrt[3]{x}}$, when x is *very great.*

XI.—CONTINUED FRACTIONS.

1. To convert any given fraction $\left(\dfrac{A}{B}\right)$ into a continued fraction of the form $a+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}$, all the letters representing positive integers.

Divide as in finding the G.C.M. of A, B. Thus

$$\begin{array}{l} B)A(a \\ \quad \underline{aB} \\ \quad\; C)B(b \\ \qquad \underline{bC} \\ \qquad\; D)C(c \\ \qquad\quad \underline{cD} \\ \qquad\quad \text{etc.} \end{array}$$

Thus $\dfrac{A}{B}=a+\dfrac{C}{B}=a+\dfrac{1}{\frac{B}{C}}$,

$\dfrac{B}{C}=b+\dfrac{D}{C}=b+\dfrac{1}{\frac{C}{D}}$; etc.

Therefore $\dfrac{A}{B}=a+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}$, or, as it may be more conveniently written, $a+\dfrac{1}{b}\,\dfrac{1}{+c+\text{etc.}}$ Here a, b, c . . . are called the successive *quotients;* the *convergents* to the value of $\dfrac{A}{B}$ are found by stopping at one of the quotients and neglecting all that follow. Thus the first three convergents are $\dfrac{a}{1}$; $a+\dfrac{1}{b}=\dfrac{ab+1}{b}$; and $a+\cfrac{1}{b+\cfrac{1}{c}}=\dfrac{abc+c+a}{bc+1}$.

2. Law of formation of successive convergents.

Let $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}, \frac{p'''}{q'''}$ be any four successive convergents, and m'', m''' the quotients corresponding to the last two of them : *assume* that

$$p''=m''p'+p \text{ and } q''=m''q'+q \ldots (a).$$

Then since

$$\frac{p''}{q''}=a+\frac{1}{b+}\;\frac{1}{c+\ldots+m''}, \text{ and } \frac{p'''}{q'''}=a+\frac{1}{b+}\;\frac{1}{c+\ldots+\left(m''+\frac{1}{m'''}\right)}.$$

$\therefore \frac{p'''}{q'''}$ is got from $\frac{p''}{q''}$ by writing $\left(m''+\frac{1}{m'''}\right)$ for m'' : but $\frac{p''}{q''}=\frac{m''p'+p}{m''q'+q}$,

$$\therefore \frac{p'''}{q'''}=\frac{\left(m''+\frac{1}{m'''}\right)p'+p}{\left(m''+\frac{1}{m'''}\right)q'+q}=\frac{m'''(m''p'+p)+p'}{m'''(m''q'+q)+q'}=\frac{m'''p''+p'}{m'''q''+q'} \text{ by } (a).$$

Therefore if the law of (a) is true for p'', q'', it is true also for p''', q'''; but it is seen to be true for the first three convergents, viz. $\frac{a}{1}$, $\frac{ab+1}{b}$, and $\frac{abc+c+a}{bc+1}$ *i.e.* $\frac{c(ab+1)+a}{c.b+1}$, therefore by *induction* it is true *universally.*

3. The convergents are alternately $<$ and $>$ than the continued fraction $\left(\frac{A}{B}\right)$.

Using the notation of the preceding article,

$$\frac{p''}{q''}=\frac{m''p'+p}{m''q'+q}.$$

Let k be the complete value of $m''+\frac{1}{m'''+\text{etc.}}$; then $\frac{A}{B}=\frac{kp'+p}{kq'+q}$;

$$\therefore \frac{A}{B}-\frac{p}{q}=\frac{kp'+p}{kq'+q}-\frac{p}{q}=\frac{k(p'q-pq')}{(kq'+q)q};$$

$$\text{and } \frac{A}{B}-\frac{p'}{q'}=\frac{kp'+p}{kq'+q}-\frac{p'}{q'}=\frac{pq'-p'q}{(kq'+q)q'};$$

and k, q, q' are all *positive* $\therefore \frac{A}{B}-\frac{p}{q}$ and $\frac{A}{B}-\frac{p'}{q'}$ are of opposite signs.

Therefore if $\frac{p}{q}$ is $<\frac{A}{B}$, then $\frac{p'}{q'}$ is $>\frac{A}{B}$, and *vice versa; i.e.* the convergents are alternately $<$ and $>$ than the continued fraction; and $\because \frac{a}{1}$ is clearly $<\frac{A}{B}$ $\therefore$ those in the even places are too great and those in the odd places too small.

4. Every Convergent is nearer to the value of the Continued Fraction than the one before it.

For, by the preceding article

$$\frac{A}{B}-\frac{p}{q}=\frac{k(p'q-pq')}{(kq'+q)q} \text{ and } \frac{p'}{q'}-\frac{A}{B}=\frac{p'q-pq'}{(kq'+q)q'}.$$

Now q' is $>q$, and k is necessarily >1.

$\therefore$ for both reasons $\frac{A}{B}-\frac{p}{q}$ is numerically $>\frac{p'}{q'}-\frac{A}{B}$, that is, $\frac{p'}{q'}$ is nearer to $\frac{A}{B}$ than $\frac{p}{q}$ is.

This article gives us the reason for the term *Convergent.*

5. To prove that $\frac{p''}{q''}-\frac{p'}{q'}=\frac{p''q'-p'q''}{q''q'}=\pm\frac{1}{q''q'}$; where $\frac{p'}{q'}$, $\frac{p''}{q''}$ are any two successive convergents.

By § 2 we have $\frac{p''}{q''}-\frac{p'}{q'}=\frac{(m''p'+p)q'-p'(m''q'+q)}{q'q''}=\frac{pq'-p'q}{q'q''}$;

$$\text{also } \frac{p''}{q''}-\frac{p'}{q'}=\frac{p''q'-p'q''}{q'q''};$$

$$\therefore p''q'-p'q''=-(p'q-pq').$$

If therefore $p'q-pq'=\pm 1$, then $p''q'-p'q''=\mp 1$.

Now by actual reduction (*second Convergent* − *first Convergent*) $=\frac{ab+1}{b}-\frac{a}{1}=\frac{+1}{b}$. Therefore *by induction* the theorem is true for *any* two successive convergents, and the sign will be $+$ or $-$, according as the first of the two is odd or even.

Hence any Convergent, as $\frac{p}{q}$, is in its lowest terms, for if not the G.C.M. of p and q would divide $p'q-pq'$, *i.e.* ± 1, which is absurd. [See § 2, p. 7.]

Again $\frac{A}{B} \backsim \frac{p'}{q'} = \frac{1}{(kq'+q)q'}$, and $\because k$ is $>1 \therefore \frac{1}{(kq'+q)q'}$ is $< \left(\frac{1}{q'}\right)^2$.

Therefore the error in taking the Convergent $\frac{p'}{q'}$, instead of the true value of $\frac{A}{B}$, is $<$ the square of $\frac{1}{q'}$.

6. *Example.* Convert $\frac{83}{68}$ into a continued fraction, and find all the convergents.

$$\begin{array}{l}
68)83(1 \\
\quad 68 \\
\quad \overline{15})68(4 \\
\qquad 60 \\
\qquad \overline{\ 8})15(1 \\
\qquad\quad 8 \\
\qquad\quad \overline{7})8(1 \\
\qquad\qquad 7 \\
\qquad\qquad \overline{1})7(7 \\
\qquad\qquad\quad \underline{7} \\
\qquad\qquad\quad \ldots
\end{array}$$

$\therefore \frac{83}{68} = 1 + \frac{1}{4}\,\frac{1}{+1+1+7}$,

$$\begin{array}{lll}
\therefore 1^{st} & \text{convergent} & = 1, \\
2^{d} & \quad " & = \frac{5}{4}, \\
3^{d} & \quad " & = \frac{1\times 5+1}{1\times 4+1} = \frac{6}{5}, \\
4^{th} & \quad " & = \frac{1\times 6+5}{1\times 5+4} = \frac{11}{9}, \\
5^{th} & \quad " & = \frac{7\times 11+6}{7\times 9+5} = \frac{83}{68}.
\end{array}$$

7. A *Quadratic Surd* may be expressed as a *recurring* continued fraction.

Example. $\sqrt{7} = 2 + \sqrt{7} - 2 = 2 + \frac{3}{\sqrt{7}+2}$

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{6}{3(\sqrt{7}+1)}$$

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{3}{\sqrt{7}+1}$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} = 1 + \frac{1}{\sqrt{7}+2}$$

$$\sqrt{7}+2 = 4 + \sqrt{7} - 2 = 4 + \frac{3}{\sqrt{7}+2}$$

$$\therefore \sqrt{7} = 2 + \frac{1}{1+}\,\frac{1}{1+1+4+1+1+1+4+\text{etc.}}$$

We may see the reason of the above process by comparing it with that for converting an ordinary vulgar fraction into a continued fraction. For

$\because \sqrt{7}$ is > 2 and < 3, we each time subtract the largest integer from what remains, which is exactly equivalent to division in the case of the ordinary fraction.

8. Every recurring continued fraction is equal to one of the roots of a quadratic equation.

Let $x = a + \frac{1}{b+}\ldots\frac{1}{+k+y}$, where y is the part which recurs and is equal to $c + \frac{1}{d+}\ldots\frac{1}{+l+y}$.

Let $\frac{p}{q}, \frac{p'}{q'}$ be successive convergents to x, the last quotient used being k,

$$\text{then } x = \frac{p'y+p}{q'y+q}:$$

and let $\frac{P}{Q}, \frac{P'}{Q'}$ be successive convergents to y, the last quotient used being l,

$$\text{then } y = \frac{P'y+P}{Q'y+Q}:$$

$$\therefore\ (xq'-p')y = p - xq \quad \therefore\ y = -\frac{xq-p}{xq'-p'}; \text{ and } Q'y^2 + (Q-P')y - P = 0.$$

Therefore by substitution,

$$Q'\left(\frac{xq-p}{xq'-p'}\right)^2 - (Q-P')\frac{xq-p}{xq'-p'} - P = 0,$$

a quadratic equation to determine x, one root of which is equal to the given recurring fraction.

9. *Example.* Let $x = 2 + \frac{1}{1+}\ \frac{1}{1+1+4+1+1+1+4+\text{etc.}}$.

$$\text{Then } x - 2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + x - 2}}}} = \frac{2x+5}{3x+8} \text{ when simplified,}$$

$$\therefore\ 3x^2 + 2x - 16 = 2x + 5,$$

$$\therefore\ 3x^2 = 21 \quad \therefore\ x = \pm\sqrt{7} \quad \therefore\ 2 + \frac{1}{1+\text{etc.}} = \sqrt{7}.$$

EXAMPLES. XI.

1. Convert the following fractions into continued fractions, and find all the convergents :—

$$\tfrac{57}{13};\ \tfrac{55}{117};\ \tfrac{479}{6628};\ \tfrac{3370}{399}.$$

2. Find the first six convergents to the values of the following surds :—

$$\sqrt{2};\ \sqrt{5};\ \sqrt{47};\ \sqrt{14}.$$

3. Express $\dfrac{a^3+6a^2+13a+10}{a^4+6a^3+14a^2+15a+7}$ as a continued fraction, and find all the convergents.

4. Solve $\dfrac{x}{b}=\cfrac{x}{a+\cfrac{x}{a+\text{etc. to }\infty}}$

5. Prove that

$$\left(1+\frac{1}{2+}\,\frac{1}{2+\ldots}\right)\left(1+\frac{1}{1+}\,\frac{1}{2+1+2+\ldots}\right)=2+\frac{1}{2+}\,\frac{1}{4+2+4+\ldots}.$$

6. Find the value of $\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\text{etc. to }\infty}}}$

7. Find the value of $1+\dfrac{1}{3+}\,\dfrac{1}{2+3+2+\ldots}$ in the form of a quadratic surd.

XII.—CONVERGENCE AND DIVERGENCE OF SERIES—INDETERMINATE CO-EFFICIENTS, AND PARTIAL FRACTIONS.

1. A series is said to be *Divergent* if the sum (S) of its terms can be made to exceed numerically any finite quantity, however large, by taking enough terms; *i.e.* if its sum has no finite *limit*.

e.g. $1+2+3+\ldots$ is *divergent*, for $S=\frac{n(n+1)}{2}$, which we can make as large as we please by increasing n.

A series is said to be *Convergent* if the sum (S) of its terms cannot be made to exceed numerically a finite quantity, however large, however many terms be taken; *i.e.* if its sum has a finite *limit*.

e.g. $1+\frac{1}{2}+\frac{1}{4}+\ldots$ is *convergent*, for $S=\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$ which is always < 2.

2. The following simple rules will apply in many cases :—

(i.) If all the terms of a series are of the same sign, and each term is greater than any finite quantity or than the preceding term, it is evident that S increases indefinitely with the number of terms, and therefore that the series is *divergent.*

(ii.) If the terms are alternately + and −, and each less than the one before, the series is *convergent.* For it can be arranged in either of the forms $(a-b)+(c-d)+(e-f)+\ldots$, or $a-(b-c)-(d-e)-\ldots$, the first of which proves the series to be positive, and the second proves it to be $<a$.

(iii.) If the ratio of each term to the one before it is numerically less than some quantity which is itself less than 1, the series is *convergent.* For if the ratio of each term to the preceding be $< r$, then $a+b+c+\ldots$ is $< a+ar+ar^2+\ldots$

$$\therefore S \text{ is } < \frac{a(1-r^n)}{1-r}, \text{ but } r \text{ is } <1.$$

$$\therefore S \text{ is } < \frac{a}{1-r}.$$

(iv.) If the ratio of each term to the one before it is equal to or greater than 1, the series is *divergent.* This is merely another way of stating (i.)

(v.) In some cases we can compare the given series with a known series, and thus determine whether it is convergent or divergent by the argument *a fortiori.*

Example.—To prove that the expansion of $(1+x)^n$ by the Binomial Theorem is *convergent* if x is < 1, and n any positive fraction.

The $(r+1)^{th}$ term of $(1+x)^n$ is $\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r$;

$\therefore$ the ratio of the $(r+1)^{th}$ term to the r^{th} term is

$$\frac{n-r+1}{r}x = -\left(1-\frac{n+1}{r}\right).x.$$

And after r becomes $> n+1$, $\frac{n+1}{r}$ is a proper fraction;

$$\therefore 1 - \frac{n+1}{r} \text{ is } < 1, \therefore \text{ the series is convergent by Rule (iii.)}$$

3. (i.) Let $a+bx+cx^2+dx^3+\ .\ .\ .\ (=S)$ be a series which is convergent for all finite values of x; then when $x=0$, S will reduce to its first term a.

For since $a+bx+cx^2+\ .\ .\ .$ is convergent, therefore the series

$$b+cx+dx^2+\ .\ .\ .\ (=A)$$
$$c+dx+ex^2+\ .\ .\ .\ (=B)$$
$$\text{etc. etc. . . .}$$

will also be convergent. [See § 1.]
Now in $S=a+x.A$ put $x=0$, then $\because A$ is finite, $\therefore xA=0$, and $\therefore S=a$.

(ii). Let the convergent series

$$S=a+bx+cx^2+dx^3+\ .\ .\ .=0 \text{ for all finite values of } x,$$
$$\text{then will } a=0,\ b=0,\ c=0.\ .\ .\ .$$

In S put $x=0$, then $bx+cx^2+\ .\ .\ .\ =x.A=0.A=0$ by (i.) for A is finite,

$$\therefore a=0, \text{ and } \therefore bx+cx^2+dx^3+\ .\ .\ .\ =0$$

for all values of x. Thus $(b+cx+dx^2+\ .\ .\ .\ )x$ vanishes for all finite values of x, however small, $\therefore b+cx+dx^2+.\ .\ .$ *i.e.* $b+x.B=0$ for all finite values of x, and therefore when x is indefinitely diminished. Therefore, as before, $b=0$, and so in the same way $c=0$, etc.

(iii.) Next, let $a+bx+cx^2+\ .\ .\ .\ =\alpha+\beta x+\gamma x^2+\ .\ .\ .$ for all finite values of x, the two series being convergent; then will $a=\alpha$, $b=\beta$, $c=\gamma$. . . .

For by transposing the terms we get

$$(a-\alpha)+(b-\beta)x+(c-\gamma)x^2+\ .\ .\ .\ =0 \text{ for all finite values of } x,$$

$$\therefore \text{by (ii.) } a-\alpha=0,\ b-\beta=0,\ c-\gamma=0\ .\ .\ .\quad \therefore a=\alpha,\ b=\beta,\ c=\gamma.\ .\ .\ .$$

This method is called *Equating co-efficients of Powers of x:* its application will be shown in the following articles.

4. Assuming that $\frac{1}{1-x+x^2}=A+A_1x+A_2x^2+\ .\ .\ .$ find the values of A, A_1, A_2 . . .

Multiply both sides by $1-x+x^2$; thus

$$\begin{aligned}1=A+A_1x+A_2x^2+A_3x^3+&\ \ldots\\ -Ax-A_1x^2-A_2x^3-&\ \ldots\\ +Ax^2+A_1x^3+&\ \ldots\end{aligned}$$

Equate co-efficients: thus $A=1$; $A_1-A=0$, $\therefore A_1=1$;
$A_2-A_1+A=0$, $\therefore A_2=0$; $A_3-A_2+A_1=0$, $\therefore A_3=-1$, etc.

$$\therefore \frac{1}{1-x+x^2}=1+x-x^3-x^4+x^6+x^7-\ \ldots$$

This is called the method of *Indeterminate Co-efficients.*

5. To resolve a fraction into its equivalent *Partial Fractions.*

The general form of the result may be shown to be as follows :—

Let $\dfrac{Ax^n+Bx^{n-1}+\ \ldots\ +K}{(x+a)^m(x^2+bx+c)}$ be any fraction.

Assume that it is equal to $Px^{n-\overline{m+2}}+\ \ldots\ +Q$

$$+\frac{R}{(x+a)^m}+\frac{S}{(x+a)^{m-1}}+\ \ldots\ +\frac{T}{x+a}+\frac{Ux+V}{x^2+bx+c}.$$

for all values of x.

Multiply up by the denominator of given fraction and by equating co-efficients, or, assuming values of x, obtain equations which give the unknown quantities $P, Q\ \ldots\ V$ in terms of the known quantities $AB\ \ldots\ K.a.b.c.$

Example: Resolve $\dfrac{3x+5}{x^2-7x-8}$ into partial fractions.

Let $\dfrac{3x+5}{x^2-7x-8}=\dfrac{A}{x+1}+\dfrac{B}{x-8}$. Multiply by x^2-7x-8,

$$\therefore 3x+5=A(x-8)+B(x+1)=(A+B)x-8A+B,$$

$$\therefore A+B=3 \text{ and } -8A+B=5 \text{ by (2)}, \therefore A=-\tfrac{2}{9} \text{ and } B=\tfrac{29}{9},$$

$$\therefore \frac{3x+5}{x^2-7x-8}=-\frac{2}{9(x+1)}+\frac{29}{9(x-8)}.$$

Otherwise, we might put $x=8$: then $24+5=A(8-8)+B(8+1)$
$\therefore B=\dfrac{29}{9}$; and by putting $x=-1$, we get $A=-\dfrac{2}{9}$ as before.

6. Thus by Partial Fractions and the Binomial Theorem we may expand complex fractions.

e.g. $$\frac{3x+5}{x^2-7x-8} = -\frac{2}{9(x+1)} + \frac{29}{9(x-8)} = -\frac{2}{9(1+x)} - \frac{29}{72\left(1-\frac{x}{8}\right)}$$

$$= -\frac{2}{9}(1+x)^{-1} - \frac{29}{72}\left(1-\frac{x}{8}\right)^{-1}$$

$$= -\frac{2}{9}(1-x+x^2-x^3+\ \ldots\ ) - \frac{29}{72}\left(1+\frac{x}{8}+\frac{x^2}{8^2}+\frac{x^3}{8^3}\ \ldots\right).$$

EXAMPLES. XII.

1. Expand $\frac{1+2x}{1-x-x^2}$ and $\frac{1-3x+2x^2}{1+x+x^2}$ each to 4 terms by the method of Indeterminate Co-efficients.

2. If $\frac{1}{(x+a)(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$, find A, B, and C.

3. Resolve into Partial Fractions $\frac{x}{x^2+6x+8}$, $\frac{x+c}{(x-a)(x-b)}$ and $\frac{x-3}{x(x+1)^2}$.

4. Resolve into Partial Fractions the expressions $\frac{4}{(x-1)(x-3)}$ and $\frac{x+3}{(x-1)(x^2+3)}$, and expand each in powers of x, obtaining in each case the co-efficient of x^r.

5. If $\frac{x^2-7x+13}{x^2-7x+12}$ be expanded in ascending powers of x, find the co-efficient of x^2 and of x^r.

6. Expand $\frac{4x-6}{x^3-6x^2+11x-6}$ in a series of ascending powers of x, and find the co-efficient of x^n.

7. Resolve $\frac{x^2-12x+13}{(x^2-5x+6)(x^2-1)}$ and $\frac{1-x-4x^2}{x-4x^2+3x^3}$ into their equivalent partial fractions.

8. Expand $\frac{1-x}{1-2x-3x^2}$ to 4 terms by the method of Indeterminate Co-efficients.

9. Resolve $\dfrac{1}{a^4 - x^4}$ and $\dfrac{7x+8}{(x^2+1)(x+1)^2}$ into partial fractions.

10. If $\dfrac{a-bx}{a+cx} = A + Bx + Cx^2 + Dx^3 + \ldots$ find the values of A, B, C, D.

11. Develope $(1-ax)^{-1}(1-bx)^{-1}$ in a series of ascending powers of x, and find the sum of n terms of the series.

12. If x is < 1, prove that the series $1 + 2x + 3x^2 + \ldots$ is convergent.

13. Prove that the series $\dfrac{1}{1} + \dfrac{1}{1.2} + \dfrac{1}{1.2.3} + \ldots$ is convergent.

XIII.—SUMMATION OF SERIES.

1. To find the sum of the *squares* of any quantities in A. P.

Let $S = a^2 + (a+d)^2 + \ldots + (a + \overline{n-1}\,.\,d)^2$.

We have $(a+d)^3 = a^3 + 3a^2d + 3ad^2 + d^3$,

$$\therefore\ (a+d)^3 - a^3 = 3a^2d + 3ad^2 + d^3.$$

Similarly, $(a+2d)^3 - (a+d)^3 = 3(a+d)^2\,.\,d + 3(a+d)d^2 + d^3$,

$$(a+3d)^3 - (a+2d)^3 = 3(a+2d)^2d + 3(a+2d)d^2 + d^3,$$

$$\ldots\ldots = \ldots\ldots$$

$$(a+nd)^3 - (a+\overline{n-1}d)^3 = 3(a+\overline{n-1}\,.\,d)^2d + 3(a+\overline{n-1}d)d^2 + d^3,$$

$\therefore$ by addition $(a+nd)^3 - a^3 = 3Sd + \dfrac{3n}{2}\{2a + \overline{n-1}d\}d^2 + nd^3$,

$$\therefore\ S = \frac{1}{3d}\left\{3a^2nd + 3an^2d^2 + n^3d^3 - \frac{3n}{2}(2a + \overline{n-1}d)d^2 - nd^3\right\}.$$

2. Thus we can get the sum of the *squares* of the first n *natural numbers, i.e.* $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

In this case $2^3 - 1^3 = 3\,.\,1^2 + 3\,.\,1 + 1$,

$$3^3 - 2^3 = 3\,.\,2^2 + 3\,.\,2 + 1,$$

$$\ldots = \ldots\ldots$$

$$(n+1)^3 - n^3 = 3\,.\,n^2 + 3\,.\,n + 1,$$

and $\therefore\ (n+1)^3 - 1^3 = 3\,.\,S + \dfrac{3n}{2}(n+1) + n$.

$$\therefore\ 6S = 2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n = n(2n^2 + 3n + 1) = n(n+1)(2n+1).$$

3. To find the sum of the *Cubes* of the first n *natural numbers*.

Let $S=1^3+2^3+\ldots+n^3$.

$$(n+1)^4=n^4+4n^3+6n^2+4n+1,$$

$$\therefore\ 2^4-1^4=4.1^3+6.1^2+4.1+1,$$

$$3^4-2^4=4.2^3+6.2^2+4.2+1,$$

$$\ldots=\ldots\ldots$$

$$(n+1)^4-n^4=4.n^3+6.n^2+4.n+1.$$

$$\therefore\ (n+1)^4-1=4S+6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}+n,$$

$$\therefore\ S=\frac{1}{4}\{n^4+4n^3+6n^2+4n-n(n+1)(2n+1)-2n(n+1)-n\}$$

$$=\frac{1}{4}\{n^4+4n^3+6n^2+3n-2n^3-3n^2-n-2n^2-2n\}$$

$$=\frac{1}{4}(n^4+2n^3+n^2)=\left\{\frac{n(n+1)}{2}\right\}^2.$$

$$\therefore\ 1^3+2^3+3^3+\ldots+n^3=\{1+2+3+\ldots+n\}^2.$$

4. *Summation of a Series by* PARTIAL FRACTIONS.

(i.) To sum $\dfrac{1}{a(a+b)}+\dfrac{1}{(a+b)(a+2b)}+\ldots$ to n terms.

$$\frac{1}{a(a+b)}=\frac{1}{b}\left\{\frac{1}{a}-\frac{1}{a+b}\right\}$$

$$\frac{1}{(a+b)(a+2b)}=\frac{1}{b}\left\{\frac{1}{a+b}-\frac{1}{a+2b}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{(a+\overline{n-1}b)(a+nb)}=\frac{1}{b}\left\{\frac{1}{a+(n-1)b}-\frac{1}{a+nb}\right\}$$

$$\therefore\ \text{by addition } S=\frac{1}{b}\left\{\frac{1}{a}-\frac{1}{a+nb}\right\}=\frac{n}{a(a+nb)}:$$

and the sum to ∞ $=\dfrac{1}{ab}$.

(ii.) Sum $\dfrac{1}{2.5}+\dfrac{1}{5.8}+\dfrac{1}{8.11}+\ldots$ to n terms and to ∞.

$$\frac{1}{2.5}=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right),$$

$$\frac{1}{5.8}=\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right), \text{ etc.} \ldots$$

$$\therefore \text{ by addition } S \text{ to } n \text{ terms}=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{2+3n}\right)=\frac{n}{2(2+3n)},$$

$$\text{and } S \text{ to } \infty=\frac{1}{6}.$$

5. *Summation by* MULTIPLICATION.

(i.) Sum $1+3.3+5.3^2+\ldots$ to n terms.

The rule in a case like this, where the series is formed by multiplying together corresponding terms of an A. P. and a G. P., is to multiply by the common ratio of the G. P. and subtract.

Thus $S=1+3.3+5.3^2+\ldots+(2n-1).3^{n-1}$,

$$\therefore 3S= \quad +1.3+3.3^2+\ldots\ldots\ldots\ldots+(2n-1).3^n,$$

$$\therefore 2S=(2n-1).3^n-1-2\{3+3^2+\ldots+3^{n-1}\}$$

$$=(2n-1).3^n-1-2\left\{\frac{3(3^{n-1}-1)}{3-1}\right\},$$

$$\therefore S=\tfrac{1}{2}\{3^n(2n-1-1)-1+3\}=3^n(n-1)+1.$$

(ii.) To sum $1+2x+3x^2+\ldots$ to n terms, and to ∞ when x is <1.

$$S=1+2x+3x^2+\ldots+n.x^{n-1},$$

$$\therefore S(1-x)=1+x+x^2+\ldots+x^{n-1}-nx^n$$

$$=\frac{1-x^n}{1-x}-nx^n,$$

$$\therefore S=\frac{1-x^n-(1-x)nx^n}{(1-x)^2},$$

And if x is <1 the sum to $\infty=\dfrac{1}{(1-x)^2}$.

(iii.) To sum $S=1+3x+6x^2+10x^3+\ldots$ to ∞.

$$\therefore S(1-x)=1+2x+3x^2+4x^3+\ldots$$

$$\therefore S(1-x)^2=1+x+x^2+x^3+\ldots$$

$$=\frac{1}{1-x}$$

$$\therefore S=\frac{1}{(1-x)^3}.$$

Notice that, to multiply the series by $(1-x)$, the first term in the new series is the same as in the old, the co-efficient of the second term in the new series is got by subtracting the first term from the co-efficient of the second in the given series, and so on.

6. *Summation by means of the general term.*

(i.) To sum $1.2+2.3+3.4+\ldots$ to n terms.

$$n^{th} \text{ term} = n(n+1) = n^2+n,$$

$$\therefore S = (1^2+2^2+\ldots+n^2)+(1+2+\ldots+n)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.$$

(ii.) To sum $1.2.3+2.3.4+3.4.5+\ldots$ to n terms.

$$n^{th} \text{ term} = n(n+1)(n+2) = n^3+3n^2+2n,$$

$$\therefore S = \left\{\frac{n(n+1)}{2}\right\}^2 + 3.\frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{4}\{n^2+n+4n+2+4\}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$ [See § 10, p. 104.]

(iii.) To sum $\frac{1.4}{2.3}+\frac{2.5}{3.4}+\ldots$ to n terms.

$$n^{th} \text{ term} = \frac{n(n+3)}{(n+1)(n+2)} = 1 - \frac{2}{(n+1)(n+2)},$$

$$\therefore S = n - 2\left\{\frac{1}{2.3}+\frac{1}{3.4}+\ldots\frac{1}{(n+1)(n+2)}\right\}$$

$$= n - 2\left\{\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n+1}-\frac{1}{n+2}\right\}$$

$$= n - 2\left\{\frac{1}{2}-\frac{1}{n+2}\right\} = n - \frac{n}{n+2} = \frac{n(n+1)}{n+2}.$$

7. To sum a series of the form

$$aa'+bb'+cc'+\ldots = S$$

where $a, b, c \ldots a', b', c' \ldots$ are the co-efficients of two known expansions $F(x)=a+bx+cx^2+\ldots$

and $F(x)=a'+b'x+c'x^2+\ldots$

Write $\frac{1}{x}$ for x in either $F(x)$ or $f(x)$.

Thus $F\left(\frac{1}{x}\right)f(x)=\left(a+\frac{b}{x}+\frac{c}{x^2}+\ldots\right)(a'+b'x+c'x^2+\ldots)$

$$=S+\text{terms containing powers of } x \text{ and of } \frac{1}{x}.$$

$\therefore$ S is the term which does not involve x in the expansion of the product $F\left(\frac{1}{x}\right)f(x)$. [See § 4, (*c.*) p. 83.]

8. To find the sum of the terms $a_0+a_1+\ldots+a_r$, where $a_0, a_1 \ldots$ are the co-efficients of a known series

$$F(x)=a_0+a_1x+a_2x^2+\ldots$$

$$\text{We have } \frac{1}{1-x}=1+x+x^2+x^3+\ldots$$

$$\therefore \frac{F(x)}{1-x}=(a_0+a_1x+\ldots)(1+x+x^2\ldots)$$

$$=a_0+(a_0+a_1)x+(a_1+a_1+a_2)x^2+\ldots$$

$\therefore a_0+a_1+a_2+\ldots+a_r$ is the co-efficient of x^r in the expansion of

$$\frac{F(x)}{1-x}.$$

e.g. Find the sum of

$$1-19+\frac{19.18}{1.2}-\frac{19.18.17}{1.2.3}+\text{ to } 11 \text{ terms.}$$

$$1-19x+\frac{19.18}{1.2}x^2\ldots=(1-x)^{19},$$

$$\therefore S=\text{co-efficient of } x^{10} \text{ in } (1-x)^{18}$$

$$=-\frac{|\underline{18}}{|\underline{8}\,|\underline{10}}=-\frac{18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8}$$

$$=-43758.$$ [See p. 81.]

9. *Recurring Series.*

A recurring series is one in which each term is equal to the sum of a constant number of the preceding terms, each multiplied by some constant quantity.

Thus in the series $1+4x+11x^2+34x^3+\ldots$

$$\left.\begin{array}{l} 11x^2=2x(4x)+3x^2(1) \\ 34x^3=2x(11x^2)+3x^2(4x) \\ \text{etc. etc.} \end{array}\right\} \begin{array}{l} \therefore 11x^2-2x(4x)-3x^2(1)=0 \\ 34x^3-2x(11x^2)-3x^2(4x)=0 \\ \text{etc. etc.} \end{array}$$

And $1-2x-3x^2$ is called the *Scale of Relation.*

To find the sum of the above series to ∞ when the n^{th} term decreases indefinitely as n increases.

$$\text{Let } S=1+4x+11x^2+34x^3+\ \ldots$$
$$\therefore -2xS= \quad -2x-\ 8x^2-22x^3\ \ldots$$
$$-3x^2S= \qquad -\ 3x^2-12x^3\ \ldots$$

$\therefore$ by addition $(1-2x-3x^2)S=1+2x$,

the co-efficient of the other powers of x vanishing because of their scale of relation :

$$\therefore S=\frac{1+2x}{1-2x-3x^2}.$$

If this fraction be split into partial fractions, and so expanded, it will be possible to find the general term, as the two series will be identical. [See § 6, p. 97.]

10. To sum to n terms the series

$$1.2\ldots r+2.3\ldots(r+1)+\ldots+n(n+1)\ldots(n+r-1).$$

Let T_n stand for the n^{th} term, then $T_{n+1}=(n+1)\ldots(n+r)$,

$$\therefore T_{n+1}=\left(\frac{n+r}{n}\right)T_n=T_n+\frac{r}{n}T_n,$$
$$\therefore n(T_{n+1}-T_n)=r\,.\,T_n.$$
$$\text{Similarly, } (n-1)(T_n-T_{n-1})=r\,.\,T_{n-1},$$
$$(n-2)(T_{n-1}-T_{n-2})=r\,.\,T_{n-2},$$
$$\ldots\ldots\ldots\ldots$$
$$1(T_2-T_1)=r\,.\,T_1,$$
$$\therefore \text{by addition } n\,.\,T_{n+1}-(T_n+\ldots+T_1)=r\,.\,S,$$
$$\therefore (r+1)S=n\,.\,T_{n+1}.$$
$$\therefore S=\frac{n(n+1)\ldots(n+r-1)(n+r)}{r+1},$$

The rule, therefore, for summing to any number of terms a series of the form given is as follows:—"Place a fresh factor at the end of the last term of the series, and divide by the number of factors so increased."

In the same way the sum of n terms of the series

$$a(a+1)\ldots(a+r-1)+(a+1)(a+2)\ldots(a+r)+\text{etc.},$$

where a is any whole number$=\{$sum of $(a+n-1)-$sum of $(a-1)\}$ terms of above series

$$=\frac{1}{r+1}\{(a+n-1)(a+n)\ldots(a+n+r-1)-(a-1)a(a+1)\ldots(a+r-1)\}.$$

11. **Given a certain number of terms of a series, it is required to find the n^{th} term, or to find the sum (S) of n terms.**

Suppose that the first four terms, T_1, T_2, T_3, T_4, are given.

Assume $T_n=A+B(n-1)+C(n-1)(n-2)+D(n-1)(n-2)(n-3)$ where the number of undetermined constants, A, B, C, D, is the same as that of the given terms.

Put $n=1, 2, 3, 4$ in succession.

Thus

$$A=T_1,\ A+B=T_2,\ A+2B+1.2C=T_3,\ A+2B+2.3C+1.2.3D=T_4.$$

And from these equations A, B, C, D may be easily determined in succession, then by substitution the n^{th} term is obtained.

[By this method we can only say that we find *a* series, of which the first few terms are given, not *the* series, for it is obvious that there may be more than one such series, since the general term is not given.]

Next, to find the sum of the first n terms.

$$\begin{aligned}S&=T_1+T_2+T_3+\ \ldots\ +T_n\\&=A\\&\quad+A+B\\&\quad+A+2B+1.2C\\&\quad+A+3B+2.3C+1.2.3D\\&\quad\ldots\ldots\ldots\ldots\ldots\ldots\\&\quad+A+(n-1)B+(n-2)(n-1)C+(n-3)(n-2)(n-1)D,\end{aligned}$$

$$\therefore S=nA+\frac{(n-1)n}{2}B+\frac{(n-2)(n-1)n}{3}C+\frac{(n-3)(n-2)(n-1)n}{4}D, \text{ by § 10.}$$

e.g. Let $S=1+4+11+\ldots$

Then $A=1$, $A+B=4$, $A+2B+2C=11$, $\therefore B=3$, $C=2$;

$$\begin{aligned}\therefore T_n&=1+3(n-1)+2(n-1)(n-2)\\&=1+3n-3+2n^2-6n+4=2n^2-3n+2.\end{aligned}$$

And $S=nA+\dfrac{(n-1)n}{2}B+\dfrac{(n-2)(n-1)n}{3}C$

$$=\frac{n}{6}\{6+9(n-1)+4(n^2-3n+2)=\frac{n(4n^2-3n+5)}{6}.$$

12. *Piles of Shot.*

(i.) To find the number of shot (S) in a complete pyramidal pile on a square base, each side of the base containing n shot. In the lowest course there will be n^2 shot, in the next above it $(n-1)^2$, and so on up to a single shot at the top.

$$\therefore S = n^2 + (n-1)^2 + \ldots + 1^2 = \frac{n(n+1)(2n+1)}{6}.$$ [See § 2, p. 99.]

(ii.) Next in a complete pile on a rectangle for its base containing $(m+n)$ and n balls in its two sides.

In the lowest course there will be $(m+n)n$, in the next above $(m+n-1)(n-1)$, and so on up to $(m+n-\overline{n-1})$, *i.e.* $(m+1)$ in the top course.

$$\begin{aligned}\therefore S &= (m+n)n + (m+n-1)(n-1) + (m+n-2)(n-2) + \ldots + (m+1)1 \\ &= m\{n + (n-1) + (n-2) + \ldots + 1\} + \{n^2 + (n-1)^2 + (n-2)^2 + \ldots + 1^2\} \\ &= \frac{mn(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3m+2n+1)}{6}.\end{aligned}$$

(iii.) In a complete triangular pile on an equilateral triangle with each side consisting of n shot.

In the lowest course there will be $\frac{n(n+1)}{2}$ shot, in the next above $\frac{(n-1)n}{2}$, and so on, up to $\frac{1.2}{2}$.

$$\begin{aligned}\therefore S &= \tfrac{1}{2}\{1.2 + 2.3 + \ldots + n(n+1)\} \\ &= \frac{1}{2}\left\{\frac{n(n+1)(n+2)}{3}\right\} = \frac{n(n+1)(n+2)}{6} \text{ by § 10.}\end{aligned}$$

(iv.) The number of shot in an incomplete pile may be found by considering it as the difference of two complete piles.

EXAMPLES. XIII.

1. Sum $1^2+3^2+5^2+\ldots$ to 12 terms.

2. Sum $2\times 5+5\times 8+8\times 11+\ldots$ to n terms and to 15 terms.

3. Assuming that $1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$, prove that the sum of the harmonic means of all the pairs of positive integers whose is equal to m will be $\frac{m^2-1}{3}$.

4. Sum (i.) $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+$etc., . . . to n terms;

and (ii.) $4+\frac{5}{2}+\frac{4}{2^2}+\frac{5}{2^3}+\frac{4}{2^4}+\ \ldots$ to ∞.

5. Find the sum of n terms of the series whose r^{th} term is $\frac{r(r+1)}{3^r}$.

6. Sum $2^2+4^2+6^2+8^2+\ \ldots$ to 10 terms and to n terms.

7. Find the number of shot in each of two rectangular complete piles containing—(i.) 16 and 7 shot, and (ii.) 35 and 15 shot respectively in the unequal sides of their bases.

8. An incomplete pyramidal pile of shot on a square base, with 25 shot in each side, consists of 10 courses, how many shot are there altogether?

9. Sum $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\ \ldots$ to n terms and to ∞.

10. Find the sum of n terms of the series of which the r^{th} term is $(2r+1)2^r$.

11. Sum $1.3.5+2.4.6+3.5.7+\ \ldots$ to n terms and to 20 terms.

12. Sum $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ \ldots$ to 12 terms and to ∞.

13. How many shot are there in each of two complete pyramidal piles on triangular bases which contain—(i.) 8 shot, and (ii.) 19 shot in each side respectively?

14. Two complete pyramidal piles are made on square bases containing 17 and 24 shot respectively in their sides; find the ratio of the number of shot in the first to that in the second.

15. Sum $3^2+6^2+9^2+12^2+\ \ldots$ to 21 terms.

16. Sum $1.2.3.4+2.3.4.5+\ldots$ to 10 terms.

17. If n be an *even* number, prove that

$$1+2^2+3+4^2+\ldots=\frac{n(n+4)(2n+1)}{12}.$$

18. Sum $\frac{1}{1.6}+\frac{1}{6.11}+\ldots$ to n terms and to ∞.

19. Write down the n^{th} term of the series $1+3+6+10+15+\ldots$ and prove that the sum of the first n terms $=\dfrac{n(n+1)(n+2)}{6}$.

20. Show that a complete square pile of shot is equivalent to two complete triangular piles.

21. Prove that

$$1\,.\,n+2(n-1)+3(n-2)+\ldots+n\,.\,1=\frac{n(n+1)(n+2)}{6}.$$

22. Sum $1+\dfrac{2}{2}+\dfrac{3}{2^2}+\dfrac{4}{2^3}+\ldots$ to ∞.

23. Find the n^{th} term and the sum of the first 15 terms of the series $1+7+21+\ldots$

24. The first two terms of a recurring series are 2 and $5x$, and the *scale of relation* is $1-3x-2x^2$; find the next four terms.

25. If in the recurring series $T_1+T_2+\ldots+T_{n-1}+T_n$ the scale of relation be $1-a-b$, find the sum in terms of a, b, T_1, T_2, T_3, T_{n-2}, T_{n-1}, and T_n.

26. Find the n^{th} term of the series $1+4x+11x^2+34x^3+\ldots$

XIV.—INDETERMINATE EQUATIONS.

1. The equation to be solved is of the form $ax\pm by=c$, where a, b, c are positive integers.

Positive integral values only of x and y are taken, and as a rule *zero* values are inadmissible; and since the equation cannot be solved if a, b have a common divisor which does not divide c, we shall assume the equation to be so simplified that a, b have no common divisor.

2. If *one* solution of $ax+by=c$ be $x=\alpha$, $y=\beta$, the *general solution* is $x=\alpha+bt$, $y=\beta-at$, where t is 0, or any integer not greater than $\dfrac{\beta}{a}$ or less than $-\dfrac{\alpha}{b}$.

For $\because$ α, β is a solution, $\therefore$ $a\alpha+b\beta=c=ax+by$,

$\therefore$ $a(x-\alpha)=b(\beta-y)$ $\therefore$ $\dfrac{x-\alpha}{\beta-y}=\dfrac{b}{a}$: but $\dfrac{b}{a}$ is in its lowest terms,

$\therefore\ x-a=bt$, and $\beta-y=at$ where t is a positive or negative integer, $\therefore$ the *general solution* is $x=a+bt$ and $y=\beta-at$, where t is 0, or any positive integer not $>\frac{\beta}{a}$ or any negative integer not numerically $>\frac{a}{b}$ for x, *i.e.* $b\left(\frac{a}{b}+t\right)$ and y, *i.e.* $a\left(\frac{\beta}{a}-t\right)$ must be positive integers by supposition. In this case, therefore, the number of solutions is *limited*.

It will be seen that the values of x and y form two A. P.'s, of which the common differences are b and $-a$ respectively.

3. (i.) If *one* solution of $ax-by=c$ be $x=a$, $y=\beta$, the *general solution* is $x=a+bt$, $y=\beta+at$ where t is 0, or any integer not less than either $-\frac{a}{b}$ or $-\frac{\beta}{a}$.

As in § 2, $a(x-a)=b(y-\beta)\quad\therefore\ \dfrac{x-a}{y-\beta}=\dfrac{b}{a}$,

and $\therefore\ x=a+bt$, $y=\beta+at$ when t is 0, or any positive integer whatever, or any negative integer not numerically $>$ either $\frac{a}{b}$ or $\frac{\beta}{a}$.

In this case, therefore, the number of solutions is *unlimited*.

Again the values of x and y form two A. P.'s of which the common differences are b and a respectively.

(ii.) A solution of $ax-by=c$ can always be found.

Let $\frac{b}{a}$ be converted into a continued fraction, and let the last two convergents be $\frac{p}{q}$ and $\frac{b}{a}$, where p, q are positive integers, and p is $<b$ and q is $<a$: then by § 5, p. 91, $ap-bq=\pm 1$.

First, let $ap-bq=+1$, then $a.pc-b.qc=c$, and $\therefore\ x=pc$, $y=qc$ is a solution.

Next, let $ap-bq=-1$, then $ab-ap-ab+bq=1$,

$$\therefore\ a(bc-pc)-b(ac-qc)=c,$$

$$\therefore\ x=(b-p)c,\ y=(a-q)c \text{ is a solution.}$$

4. *Examples.*

(i.) Find all the solutions of $5x+17y=161$.

Divide by the least co-efficient: thus

$$x+3y+\frac{2y-1}{5}=32\quad\therefore\ \frac{2y-1}{5}=\text{integer.}$$

Multiply the numerator by some integer such that when the product is divided by the denominator the remainder will be $y \pm$ a number. Thus

$$\frac{6y-3}{5} = y + \frac{y-3}{5} = \text{integer} \quad \therefore \frac{y-3}{5} = t \text{ (an integer or 0)},$$

$$\therefore y = 3 + 5t, \text{ and } \therefore \text{ by substitution } x + 9 + 15t + \frac{6 + 10t - 1}{5} = 32,$$

$$\therefore x = 22 - 17t,$$

$\therefore t = 0$ and 1 are the only admissible values, and then $x = 22$, 5 and $y = 3$, 8.

(ii.) Find the least solution of $6x - 23y = 20$.

$$x - 4y + \frac{y-2}{6} = 3, \quad \therefore y = 2 + 6t, \text{ and } \therefore x = 3 + 8 + 24t - t = 11 + 23t.$$

$t = 0$ gives the least solution, and then $x = 11$, $y = 2$.

(iii.) Find the number of solutions of $7x + 2y = 103$.

$$3x + \frac{x-1}{2} + y = 51, \quad \therefore x = 2t + 1, \quad \therefore y = \frac{103 - 7x}{2} = 48 - 7t,$$

$\therefore t$ may have any value from 0 to $+6$, $\therefore$ number of solutions $= 7$.

(iv.) Find the least number which, divided by 17 and 26, leaves remainders 13 and 21 respectively.

$$N = 17x + 13 = 26y + 21, \quad \therefore 17x - 26y = 8,$$

$$\therefore x - y - \frac{9y+8}{17} = 0, \quad \therefore \frac{9y+8}{17} = \text{integer},$$

$$\therefore \frac{18y+16}{17} = y + 1 + \frac{y-1}{17} = \text{integer}, \quad \therefore y = 1 + 17t,$$

$$\text{and } x = \frac{26y+8}{17} = \frac{34 + 26 \times 17t}{17} = 2 + 26t.$$

$t = 0$ gives the least solution, and then $x = 2$ and $N = 47$.

EXAMPLES. XIV.

1. Find all the solutions of the following equations :—

(α.) $5x + 8y = 29$; (β.) $y = 13 + \frac{4}{13}(15 - x)$; (γ.) $71x + 17y = 1005$.

2. How many solutions are there of the equations

(i.) $2373=13x+24y$; and (ii.) $12x=106-7y$.

3. Find the least solution and the general solution of

(i.) $17x=11y+86$; and (ii.) $15x-19y=10$.

4. Find the smallest number which, when divided by 5 or by 4, leaves remainder 1 in each case, and when divided by 3 leaves no remainder.

5. A farmer owing another £7, paid him by giving him a certain number of cows worth £20 each, and receiving from him a certain number of horses worth £31 each. How could this be most simply done?

6. A bill of £4 is paid with 80 coins, consisting of half-crowns, shillings, and fourpenny bits. If the number of shillings used be the least possible, find the numbers of each coin.

7. Find all the positive integral solutions of

(α.) $xy+x+y=35$, and of (β.) $xy-2x-y=8$.

8. Find two positive integers such that if one be multiplied by 35 and the other by 43, the sum of the products may be 4000.

9. Find the number of solutions (i.) of $5x+8y=1989$, and (ii.) of $3x+5y=173$ in positive integers.

10. In how many ways can a sum of £100 be paid in guineas and pistoles, worth 17 shillings each?

XV.—GENERAL THEORY OF EQUATIONS.

1. Every equation of the n^{th} degree in x involving only integral powers of x may be reduced to the form

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+ \;.\;.\;.\; +p_{n-1}x+p_n=0 \;.\;.\;. \quad \text{(i.)}$$

where n is a positive integer, and $p_0, p_1, p_2 \;.\;.\;.$ real quantities, positive or negative, or any one except p_0 zero.

If any one or more of the co-efficients vanish the equation is said to be *incomplete*, since some powers of x will be wanting: otherwise it is said to be *complete*.

We shall denote the left-hand member of equation (i.) by $F(x)$.

2. If a be a root of the equation

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+ \ . \ . \ . \ +p_n=0$$

then the left-hand member is divisible by $x-a$ without remainder, and *conversely*.

(i.) Suppose that $F(x)$ is divided by $x-a$ in the ordinary way: then since $x-a$ is of only one dimension in x, the division may be continued till there is a remainder R which does not contain x. Let Q be the quotient containing only positive integral powers of x, which therefore cannot become infinite for any finite value of x.

$$\therefore F(x)=(x-a)Q+R.$$

In this put $x=a$, then $F(x)=0$, for by supposition a is a root, and $(x-a)Q=(a-a)Q=0$, $\therefore R=0$: and $\because R$ does not contain x, it is not altered by the substitution of a for x, $\therefore R=0$ whatever x may be; and therefore $F(x)$ is divisible by $x-a$.

(ii.) *Conversely*, if $F(x)$ is divisible by $x-a$, then a is a root of $F(x)=0$.

For, $F(x)=(x-a)Q$ since by supposition there is no remainder, and Q contains only positive integral powers of x.

If then we put $x=a$, $x-a=0$.

$$\therefore F(a)=0, \text{ or } a \text{ is a root of } F(x)=0.$$

3. (a.) To find the quotient (Q) and the remainder (R) when $F(x)$ is divided by $x-a$.

$$\begin{aligned} F(x) &= F(x)-F(a)+F(a) \\ &= (p_0x^n+p_1x^{n-1}+ \ . \ . \ . \ +p_n)-(p_0a^n+p_1a^{n-1}+ \ . \ . \ . +p_n)+F(a) \\ &= \{p_0(x^n-a^n)+p_1(x^{n-1}-a^{n-1})+ \ . \ . \ . \ +p_{n-1}(x-a)\}+F(a). \end{aligned}$$

Divide both sides by $x-a$; then by § 10, p. 2,

$$Q=p_0(x^{n-1}+ax^{n-2}+a^2x^{n-3}+. \ . \ .+a^{n-1})+p_1(x^{n-2}+ax^{n-3}+. \ . \ . \ ) + . \ . \ . +p_{n-1}$$

$$\text{and } R=F(a)=p_0a^n+p_1a^{n-1}+ \ . \ . \ . +p_n.$$

By re-arranging the terms we get

$$Q=p_0x^{n-1}+(p_0a+p_1)x^{n-2}+(p_0a^2+p_1a+p_2)x^{n-3}+ \ . . . \ +(p_0a^{n-1}+. \ . \ .)x.$$

We see in this result that the successive co-efficients and the remainder are found by multiplying the last obtained co-efficient by a and adding to the result the next co-efficient from $F(x)$. The practical method of using this result will be seen from the following examples:—

Ex. (i.) Find the quotient and remainder when

$$3x^3+17x^2+10x-14 \text{ is divided by } x-4.$$

Write down the co-efficients only with their proper signs. Thus

$$\begin{array}{l} 3+17+\ 10-\ 14 \\ \underline{\quad 12+116+504} \\ \quad 29+126+490 \end{array} \quad \therefore\ Q=3x^2+29x+126, \text{ and } R=490.$$

Ex. (ii.) Find Q and R, when $5x^4-x^3+2x-3$ is divided by $x+2$.

$$\begin{array}{l} 5-\ 1+\ 0+\ 2-\ 3 \\ \underline{\ -10+22-44+84} \\ \ -11+22-42+81. \end{array} \quad \therefore Q=5x^3-11x^2+22x-42, \text{ and } R=81.$$

(β.) To find the numerical value of $F(x)$ when any number a is substituted for x.

This is clearly the same as finding the remainder when $F(x)$ is divided by $x-a$. [See a.]

Ex. Find the numerical value of $2x^4-x^2+3x-1$ when 3 is substituted for x.

$$\begin{array}{r} 2+0-\ 1+\ 3-\ \ 1 \\ \underline{+6+18+51+162} \\ +6+17+54+161 \end{array}$$

$\therefore$ the result of the substitution is 161.

4. Every equation of the n^{th} degree has n roots and no more.

We must *assume* in the following proof that every equation of the form $F(x)=0$ has *a* root, real or imaginary.

Let a be a root of $F(x)=0$, then by § 3 $F(x)$ is divisible by $(x-a)$. Let $F_1(x)$ be the quotient, then $F(x)=(x-a)F_1(x)$ where $F_1(x)$ is similar to $F(x)$ but of one dimension lower. Hence $F_1(x)=0$ must have a root b, and therefore $F_1(x)=(x-b)F_2(x)$ as before. Thus eventually we shall get $F(x)=p_0(x-a)(x-b)\ .\ .\ .\ (x-k)$, the product of n simple factors with a co-efficient p_0 which does not contain x. $\therefore$ $F(x)$ will vanish if we put $x=a, b, c\ .\ .\ .$ or k.

That is $a, b, c\ .\ .\ .\ k$ are n roots of $F(x)=0$.

Secondly, if possible let there be another root q of $F(x)=0$ different from any one of those already found. Then $F(q)=0$, *i.e.* by substitution $p_0(q-a)(q-b)(q-c)\ .\ .\ .\ (q-k)=0$, but not one of these factors can vanish, therefore q cannot be a root.

5. In any equation with *real* co-efficients *imaginary* roots occur in pairs.

For let $\alpha+\beta\sqrt{-1}$ be a root; then substituting for x we get

$$p_0(\alpha+\beta\sqrt{-1})^n+p_1(\alpha+\beta\sqrt{-1})^{n-1}+\ldots=0.$$

Expanding each term by the Binomial Theorem we get $P+Q\sqrt{-1}=0$; P and Q being both real. The *odd* powers of $\beta\sqrt{-1}$ will give *imaginary* terms and the *even* powers *real* terms. But $\because P+Q\sqrt{-1}=0$, $\therefore P=0$ and $Q=0$.

$\therefore P-Q\sqrt{-1}=0$. That is $p_0(\alpha-\beta\sqrt{-1})^n+p_1(\alpha-\beta\sqrt{-1})^{n-1}+\ldots=0$,

$$\therefore \alpha-\beta\sqrt{-1} \text{ is also a root.}$$

$\therefore F(x)$ contains *two imaginary* factors $x-(\alpha+\beta\sqrt{-1})$ and $x-(\alpha-\beta\sqrt{-1})$ which give *one real* quadratic factor $x^2-2\alpha x+\alpha^2+\beta^2$.

In the same way it may be shown that *irrational* roots of the form $a\pm\sqrt{b}$ occur in pairs in an equation with *rational* co-efficients.

6. Let the equation $F(x)=0$ be so reduced by division if necessary that p_0 the co-efficient of the highest power of x is equal to 1. Then if the n roots are $a, b, c \ldots k$ we have by the proof of the Binomial Theorem—

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n$$
$$=(x-a)(x-b)(x-c)\ldots(x-k)$$
$$=x^n-(a+b+c\ldots)x^{n-1}+(ab+bc\ldots)x^{n-2}\ldots+(-1)^n abc\ldots k.$$
$$\therefore -p_1=a+b+c\ldots+k=\text{sum of roots,}$$
$$p_2=ab+bc+\ldots \quad =\text{sum of each pair of the roots}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$(-1)^n p_n=abc\ldots k=\text{product of roots.}$$ [See p. 80.]

Ex. (i.) To find the sum of the *squares* of the roots.

$$a^2+b^2+\ldots+k^2=(a+b+\ldots+k)^2-2(ab+bc+\ldots)=p_1^2-2p_2.$$

Ex. (ii.) To find sum of the *reciprocals* of the roots.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots+\frac{1}{k}=\frac{bc\ldots k+ac\ldots k+\ldots+abc\ldots}{abc\ldots k}=-\frac{p_{n-1}}{p_n}.$$

7. If the signs of the alternate terms of a *complete* equation be changed, the signs of all the roots will be changed.

For, put $-y$ for x; then when n is even, equation becomes

$$p_0y^n - p_1y^{n-1} + p_2y^{n-2} \ldots = 0;$$

and when n is odd, it becomes

$$-p_0y^n + p_1y^{n-1} - p_2y^{n-2} + \ldots = 0,$$

$$i.e.\ p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \ldots = 0.$$

And $\because$ $y = -x$, the values of y are $-a, -b, -c \ldots$ if those of x are $a, b, c \ldots$ *i.e.* the signs of all the roots are changed.

8. To transform an equation into one whose roots are the *reciprocals* of the roots of the original equation.

Put $y = \frac{1}{x}$, then $x = \frac{1}{y}$, and equation (i.) becomes

$$p_0\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \ldots + p_n = 0.$$

i.e. $p_ny^n + \ldots + p_2y^2 + p_1y + p_0 = 0$; and we see that the new equation can be got from the original equation by writing the co-efficients in the reverse order.

9. To transform an equation into one whose roots are the roots of the original equation multiplied by a given quantity (k).

Put $y = kx$, then $x = \frac{y}{k}$, and equation (i.) becomes

$$p_0\left(\frac{y}{k}\right)^n + p_1\left(\frac{y}{k}\right)^{n-1} + \ldots + p_n = 0,$$

$$i.e.\ p_0y^n + p_1ky^{n-1} + \ldots + p_nk^n = 0.$$

Here k of course may be $+$ or $-$, integral or fractional.

10. To transform an equation into one whose roots are greater or less than the roots of the original equation by a given quantity (k).

Put $y = x \pm k$, then $x = y \mp k$ and equation (i.) becomes

$$p_0(y \mp k)^n + p_1(y \mp k)^{n-1} + \ldots + p_n = 0.$$

Ex. Diminish by 3 the roots of the equation

$$x^3 - 12x^2 + 47x - 60 = 0, \text{ and thus solve it.}$$

Put $y=x-3$: then $x=y+3$. Substitute: thus

$$(y+3)^3-12(y+3)^2+47(y+3)-60=0,$$
$$\therefore\ y^3-3y^2+2y=0,$$
$$\therefore\ y=0, \text{ and } y^2-3y+2=0,$$
$$\therefore\ y=0,\ 1 \text{ or } 2, \text{ and } \therefore\ x=3,\ 4 \text{ or } 5.$$

11. To get rid of any term from an equation we use the method of the preceding article and substitute $y+h$ for x, and equate the co-efficient of the term, which we wish to get rid of, to zero.

Ex. Transform the equation $x^3-12x^2+16x-56=0$ into one without a term containing the square of the unknown quantity.

Put $y+h$ for x

$$\therefore\ y^3+(3h-12)y^2+(3h^2-24h+16)y+h^3-12h^2+16h-56=0,$$
$$\therefore\ 3h-12=0, \quad \therefore\ h=4, \text{ and } \therefore \text{ the equation is } y^3-32y-120=0.$$

It will be found impossible to get rid by this method of either the *first* or *last* term of that equation and so to reduce the degree of the equation.

12. To get rid of *fractional co-efficients* from an equation.

Put $y=kx$ and substitute as in § 9, and then determine k so that the denominators may all cancel out.

Ex. (i.) Get rid of fractions from $x^3+\frac{2}{3}x^2+\frac{3}{2}x+\frac{1}{6}=0$.

$$\text{Here } y^3+\tfrac{2}{3}ky^2+\tfrac{3}{2}k^2y+\tfrac{1}{6}k^3=0,$$

$\therefore\ k=6$ and the equation becomes $y^3+4y^2+54y+36=0$, and $x=\dfrac{y}{6}$.

Ex. (ii.) Get rid of fractions from $x^3-\frac{4}{3}x^2-\frac{3}{8}x+\frac{5}{72}=0$.

$$\text{Here } y^3-\tfrac{4}{3}ky^2-\tfrac{3}{8}k^2y+\tfrac{5}{72}k^3=0$$

and $\therefore$ $8=2^3$ and $72=2^3\times 3^2$ we see that $k=2^2\times 3=12$

and $\therefore$ the equation is $y^3-16y^2-54y+120=0$, and $x=\dfrac{y}{12}$.

13. *Descartes' Rule of Signs.*

(α.) In any equation $F(x)=0$, *complete* or *incomplete*, the number of *positive* real roots cannot exceed the number of changes of sign in the co-efficients of $F(x)$; and (β.) in any *complete* equation the number of *negative* real roots cannot exceed the number of continuations of sign of the co-efficients.

(*a.*) *First*, let $+ - + + - - - + -$ (i.) represent the signs of the terms of the left-hand side of any *complete* equation. Let a new positive root a be introduced, then we shall have to multiply by a factor $x - a$.

Writing down only the signs of the operation, we have

$$\begin{array}{l} + - + + - - - + - \\ \quad - + - - + + + - + \\ \hline + - + \pm - \mp \mp + - + \quad \ldots \text{(ii.)} \end{array}$$

In the result (ii.) we write the doubtful sign where the result may be either + or − or 0. Comparing (ii.) with (i.), we see that for every continuation or group of continuations in (i.) there is a corresponding ambiguity or group of ambiguities in (ii.), the signs before and after each group being unlike, and that there is a final change of sign added at the end. Hence taking the most unfavourable case, viz., the upper signs of the ambiguities in (ii.), we have the same signs as in (i.), with the additional change of sign at the end. Therefore the new positive root has introduced at least one change of sign.

Next, let the equation be *incomplete;* then it will consist of groups of + and − terms separated by groups of 0. Now, by what has been already proved, each group (being considered as a complete equation) will introduce at least one change of sign; and each group of 0 can take away at most one change of sign; and the groups of + and − are greater by 1 than the groups of 0; therefore at least one change of sign must be introduced.

The same will be true for each of the + roots $b, c, \ldots$ and the corresponding factors $x - b, x - c \ldots$

Therefore no equation *complete* or *incomplete* can have more + roots than it has changes of sign.

(β.) In any *complete* equation $f(x) = 0$ put $-y$ for x; then $f(y)$ will have as many changes of sign as $f(x)$ has continuations, thus (i.) would become

$$+ + + - - + - - - \quad \text{[See § 7.]}$$

and by (a) the equation $f(y) = 0$ cannot have more positive roots than $f(y)$ has changes; therefore, since $x = -y$, $f(x) = 0$ cannot have more − roots than $f(x)$ has continuations.

Now in any complete equation the number of changes + number of continuations = degree of the equation = number of roots; therefore in any *complete* equation, with all its roots real, the number of positive and negative roots = number of changes and continuations of signs respectively.

Ex. $x^3+2x^2-23x-60=0$ has *one* positive and *two* negative roots. They might be found to be $+5$, -4, and -3.

Descartes' rule may be stated otherwise, as follows :—

The equation $F(x)=0$ whether *complete* or *incomplete* cannot have more *positive* real roots than $F(x)$ has changes of sign or more *negative* real roots than $F(-x)$ has changes of sign.

14. If n be a positive integer, any small change in x where x is any finite quantity produces only a small corresponding change in the value of x^n.

For let h represent the change in x, so that x becomes $x+h$; then x^n becomes

$$(x+h)^n=x^n+nhx^{n-1}+\frac{n(n-1)}{\underline{|2}}h^2x^{n-2}+\ \ldots\ +h^n,$$

the series being finite, $\because$ n is a positive integer and x is finite. Therefore if h be indefinitely small, all the terms after nhx^{n-1} will vanish in comparison with it ; $\therefore$ $(x+h)^n-x^n=hnx^{n-1}$, in the limit when h is indefinitely diminished, *i.e.* the change in $x^n \propto h$, which is the change in x. That is, the value of x^n changes continuously as x changes continuously. Therefore if x^n changes from $+$ to $-$, as x changes from a to b, then for some value of x between a and b must $x^n=0$.

If, then, we consider the series $f(x)=p_0x^n+p_1x^{n-1}+\ \ldots\ p_n$, where n is a positive integer, each term by the above will vary continuously as x varies, and therefore their sum, being finite for all finite values of x, will vary continuously. If, then, $f(x)$ change from $+$ to $-$ as x changes from a to b, it must pass through zero for some intermediate value of x.

Therefore if $f(x)$ is positive when $x=a$ and negative when $x=b$, then must $f(x)=0$ have *one real* root at least between a and b, and may have *any odd* number of *real* roots.

15. *Reciprocal or Recurring Equations.*

A reciprocal equation is one in which the roots occur in pairs, such as a, $\frac{1}{a}$; b, $\frac{1}{b}$; 1, 1 ; . . . and is therefore unaltered when $\frac{1}{x}$ is written for x.

It must $\therefore$ by § 8 be of one of the forms

$$\left.\begin{array}{l} p_0x^n \pm p_1x^{n-1} \ldots \pm p_1x + p_0 = 0 \ldots (\alpha) \\ p_0x^n \pm p_1x^{n-1} \ldots \mp p_1x - p_0 = 0 \ldots (\beta) \end{array}\right\} \text{the co-efficients of terms equidistant from the two ends being equal and of like or unlike signs.}$$

If, then, n is odd (α) is satisfied by $x = -1$, and $\therefore x + 1$ is a factor

$n \ldots (\beta) \ldots\ldots\ldots x = +1 \ldots x - 1 \ldots$

n is even $(\beta) \ldots\ldots\ldots x = \pm 1 \ldots x^2 - 1 \ldots$

Therefore in every case we can so divide down a reciprocal equation as to get it into the standard form

$$px^n + qx^{n-1} + rx^{n-2} + \ldots + tx^{\frac{n}{2}} + \ldots + rx^2 + qx + p = 0,$$

where n is *even*.

$$\therefore p(x^n + 1) + q(x^{n-1} + x) + r(x^{n-2} + x^2) + \ldots + t \, . \, x^{\frac{n}{2}} = 0.$$

Divide by $x^{\frac{n}{2}} \therefore p\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right) + q\left(x^{\frac{n}{2}-1} + \frac{1}{x^{\frac{n}{2}-1}}\right) + \ldots + t = 0.$

Now, put $x + \frac{1}{x} = z$, then it is easily proved that

$$x^{n+2} + \frac{1}{x^{n+2}} = z\left(x^{n+1} + \frac{1}{x^{n+1}}\right) - \left(x^n + \frac{1}{x^n}\right).$$

In this put $n = 0, 1, 2, 3, \ldots$ in succession;

$$\therefore x^2 + \frac{1}{x^2} = z^2 - 2, \; x^3 + \frac{1}{x^3} = z(z^2 - 2) - z = z^3 - 3z, \text{ etc.}$$

Thus by substitution the equation is reduced to one of the $\left(\frac{n}{2}\right)^{th}$ degree in z; and so $\frac{n}{2}$ values of z are obtained, and each of these gives a quadratic for x, $\because z = x + \frac{1}{x}$; and thus the n values of x are obtained.

Ex. Solve $x^5 - 4x^4 + 3x^3 + 3x^2 - 4x + 1 = 0$.

Divide out by $x + 1$; thus

$$x^4 - 5x^3 + 8x^2 - 5x + 1 = 0,$$

$$\therefore \left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 8 = 0.$$

Put $x+\dfrac{1}{x}=z$, $\therefore\ x^2+\dfrac{1}{x^2}=z^2-2$,

$$\therefore\ z^2-2-5z+8=0,\ \therefore\ z^2-5z+6=0,$$

$$\therefore\ z=2 \text{ or } 3,\ \therefore\ x+\frac{1}{x}=2 \text{ or } 3,$$

$$\therefore\ x^2-2x+1=0, \text{ or } x^2-3x=-1,$$

$$\therefore\ (x-1)^2=0, \text{ or } x^2-3x+(\tfrac{3}{2})^2=\tfrac{9}{4}-1=\tfrac{5}{4},$$

$$\therefore\ x=-1,\ +1,\ +1,\ \frac{3+\sqrt{5}}{2} \text{ and } \frac{3-\sqrt{5}}{2}.$$

16. It is observed that 3 and -1 are roots of the equation $x^4-4x^3+8x+3=0$; what are the other roots?

$$\because\ 3 \text{ and } -1 \text{ are roots.}$$

$\therefore\ (x-3),\ (x+1)$ are factors. [See § 2, p. 112.]

Divide out therefore by x^2-2x-3. The result is

$$x^2-2x-1=0,\quad \therefore\ x=1\pm\sqrt{2}.$$

17. Solve the equation $x^3-10x^2+29x-20=0$, one of the roots of which is equal to the sum of the other two.

Let p, q and $p+q$ be the roots; then the given equation is equivalent to

$$(x-p)(x-q)(x-p-q)=0,$$

$$i.e.\ x^3-2(p+q)x^2+(p^2+3pq+q^2)x-pq(p+q)=0,$$

$$\therefore\ p+q=5,\ p^2+3pq+q^2=29,\ pq(p+q)=20,\quad \therefore\ pq=4,\quad \therefore\ p-q=\pm3.$$

$$\therefore\ p=4 \text{ or } 1, \text{ and } q=1 \text{ or } 4,$$

$$\therefore\ x=1,\ 4, \text{ or } 5.$$

18. Form the equation of the fourth degree with rational coefficients, one of whose roots is $\sqrt{2}+\sqrt{-3}$.

$$(x-\sqrt{2}-\sqrt{-3})(x-\sqrt{2}+\sqrt{-3})=(x-\sqrt{2})^2+3=x^2-2\sqrt{2}x+5.$$

Again $(x^2+5-2\sqrt{2}x)(x^2+5+2\sqrt{2}x)=(x^2+5)^2-8x^2$.

$$\therefore\ \text{the required equation is } x^4+2x^2+25=0.$$

19. Transform the equation $x^3+px^2+qx+r=0$, whose roots are a, b, c into one whose roots are bc, ca, ab.

The equation whose roots are a, b, c is

$$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc=0,$$

$$\therefore\ -p=a+b+c,\quad q=bc+ca+ab, \text{ and } -r=abc.$$

Again, the equation whose roots are bc, ca, ab is

$$(x-bc)(x-ca)(x-ab)=x^3-(bc+ca+ab)x^2+abc(a+b+c)x-a^2b^2c^2=0,$$
$$i.e.\ x^3-qx^2+prx-r^2=0.$$

20. *Cardan's Solution of a Cubic Equation.*

Let the equation, if necessary, be reduced to the form $x^3+px+q=0$ [§ 11], and put $x=y+z$. Then the equation becomes by substitution

$$(3yz+p)x+y^3+z^3+q=0, \quad \because x^3=(y+z)^3=y^3+3yz(y+z)+z^3.$$

Now, as we have only assumed one relation between y and z, we may assume another, viz., $3yz+p=0$, then $yz=-\frac{p}{3}$. But $y^3+z^3=-q$.

$\therefore y^3$ and z^3 are the roots of $X^2+qX-\frac{p^3}{27}=0$, for $-q$ is their sum, and $-\frac{p^3}{27}$ their product. [See § 6, p. 21.]

$$\therefore X^2+qX+\left(\frac{q}{2}\right)^2=\frac{q^2}{4}+\frac{p^3}{27}, \quad \therefore y^3=-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}},$$
$$\text{and } z^3=-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}.$$

$$\text{Therefore } x=y+z=\left\{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}}+\left\{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}}$$

If p is positive and q positive or negative, or if p is negative and q positive or negative and $\frac{p^3}{27}$ numerically $<\frac{q^2}{4}$, then y and z are both real, for the quantity under the square root is positive. Again, if $\frac{p^3}{27}$ is $<\frac{q^2}{4}$ and negative, then $\frac{q^2}{4}+\frac{p^3}{27}$ is negative, and therefore both y and z are imaginary, but $y+z$ will be real, for $(\alpha+\beta\sqrt{-1})^{\frac{1}{3}}+(\alpha-\beta\sqrt{-1})^{\frac{1}{3}}$ is always real, as may be proved by the Binomial Theorem, the imaginary terms cancelling one another.

21. Every cubic equation of the form $x^3+px+q=0$ where p and q are positive quantities, has two impossible roots and one negative real root.

There can be no real positive root, for there is no change of sign. Again, if $-x$ be put for x, we get $-x^3-qx+r$, therefore there cannot be more than one negative real root. Therefore there must be two impossible roots and one real negative root.

22. *Example.*—Solve $x^3-9x+28=0$ by Cardan's method.

$$\frac{q^2}{4}+\frac{p^3}{27}=196-27=169,$$

$$\therefore x=\{-14+13\}^{\frac{1}{3}}+\{-14-13\}^{\frac{1}{3}}=-1-3=-4.$$

The two imaginary roots are then found to be $2\pm\sqrt{-3}$.

EXAMPLES. XV.

1. Find the equation, each of whose roots is greater by unity than a root of $x^3-5x^2+6x-3=0$.

2. Transform the equation $x^3-\frac{x}{4}-\frac{3}{4}=0$ into one whose roots exceed by $\frac{3}{2}$ the corresponding roots of the given equation.

3. Find the sum, the sum of the squares, and the sum of the cubes of the roots of the equation $x^3-px-r=0$.

4. Show that $x^3-a^3=0$ has one *real* and two *imaginary* roots, and that the product of the two latter = the square of their sum.

5. If $2(p+q+r)=a^2+b^2+c^2$, and the roots of $x^2+ax-p=0$ be b, c, and those of $x^2+bx-q=0$ be c, a, then the equation whose roots are a, b is $x^2+cx-r=0$.

6. Transform the equation $x^3-px^2+qx-r=0$ into one, the roots of which are obtained by subtracting one of the roots of the given equation from the sum of the other two.

Hence solve $32x^3+16x^2-18x-9=0$.

7. Find the condition that $x^2+ax+1=0$, and $x^3-2x^2+2x-1=0$, may have two roots in common.

8. If a, β, γ are the roots of $x^3-px+r=0$, prove that the equation whose roots are $\frac{\beta+\gamma}{a}$, $\frac{\gamma+a}{\beta}$, $\frac{a+\beta}{\gamma}$ is

$$rx^3+3rx^2+(3r-p^3)x+r^3=0.$$

9. One of the roots of the equation $x^3-13x^2+15x+189=0$ exceeds another by 2 : solve it.

10. One of the roots of $x^4+4x^3+5x^2+2x-2=0$ is $-1+\sqrt{-1}$: find the others.

11. The roots of $36x^3+72x^2+23x-6=0$ are in A. P.: find them.

12. The roots of $8x^3-14x^2+7x-1=0$ are in G. P.: find them.

13. Find an equation with integral real co-efficients one of whose roots is $\sqrt{2}+\sqrt{-1}$.

14. One root of $3x^4-10x^3+4x^2-x-6=0$ is $\dfrac{1+\sqrt{-3}}{2}$: find the others.

15. Having given that $2+\sqrt{-3}$ in one root of $x^4-4x^2+8x+35=0$: find all the roots.

16. Solve (i.) $x^3-8x^2+19x-12=0$; and (ii.) $x^4-6x^2-16x+21=0$.

17. Diminish by 3 the roots of the equation

$$x^5-4x^4+3x^2-4x+6=0.$$

18. The roots of $x^3-13x^2+39x-27=0$ are in G. P. Find them.

19. The roots of $4x^3-32x^2-x+8=0$ are of the forms $+a$, $-a$, and b. Solve the equation.

20. Two roots of $x^3-\dfrac{7}{2}x^2+\dfrac{7}{2}x-1=0$ are of the forms a and $\dfrac{1}{a}$. Find all three roots.

21. Prove that $x^5-2x^2+1=0$ has at least two imaginary roots.

22. Find the numerical value (i.) of $3x^4+x^3+9x^2-1$ when 12 is substituted for x, and (ii.) of $8x^3-7x^2-5x+6$ when -3 is substituted for x.

23. Find the quotient and remainder in each of the following cases: (i.) when $3x^3-7x+8$ is divided by $x-4$, (ii.) $2x^4+3x^3+x^2-x-2$ by $x+2$, and (iii.) $5x^3+x^2+1$ by $x+7$.

24. Find the quotient and remainder in each of the following cases: when $5x^4+7x^3+8x^2-2x-10$ is divided by (i.) $x-3$, (ii.) $x-5$, and (iii.) $x+2$.

25. Solve $x^3-3x+2=0$ by Cardan's method.

26. Find the numerical values of $5x^4-x^3-x^2+2x-1$, and $3x^4-2x^2+6x+3$ when 4 is substituted for x.

27. Get rid of the co-efficient of x^3 in the equation $2x^4+x^3+3x^2+2=0$.

28. Solve the equation $x^3+x-10=0$ by Cardan's method.

29. Show that the equation $x^6 - x^4 + x^3 + x + 1 = 0$ has at least two imaginary roots; and find the least possible number of imaginary roots of the equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$.

30. Prove that the real root of the equation $x^3 + 12x = 12$, is $2\sqrt[3]{2} - (\sqrt[3]{2})^2$.

31. Find the equation whose roots are each the cubes of the roots of the equation $x^2 - 5x + 6 = 0$, and test the accuracy of the result by the solution of the two equations.

32. Solve the reciprocal equation

$$x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$$

33. Solve $6x^3 - 23x^2 - 6x + 8 = 0$, having given that the roots are in H. P.

34. Two roots of $x^4 + x^3 - 11x^2 - 9x + 18 = 0$ are of the form $+a$, $-a$: find all the roots.

35. Form the biquadratic equation with real and rational co-efficients, two roots of which are $1 + \sqrt{a^3}$ and $-\sqrt{-b}$.

36. One root of $x^4 + x^3 - 8x^2 - 16x - 8 = 0$ is $1 - \sqrt{5}$; find the others.

37. The equation $x^3 - 5x^2 + 8x - 4 = 0$ has two equal roots: find all.

38. Solve $x^3 - 11x^2 + 36x - 36 = 0$, the roots being in H. P.

39. Prove that $x^4 - x^3 - x^2 + 2x - 3 = 0$ has a real root between 1 and 2.

40. Prove that every equation of an odd degree has at least one real root with a sign contrary to that of the last term; and every equation of an even degree with its last term negative has at least two real roots of opposite signs.

XVI.—MISCELLANEOUS PROBLEMS.

(i.) If $a + b + c = 0$, then $a\dfrac{b^3 - c^3}{b - c} + b\dfrac{c^3 - a^3}{c - a} + c\dfrac{a^3 - b^3}{a - b} = 0.$

For $a\dfrac{b^3 - c^3}{b - c} + \ldots + \ldots = a(b^2 + bc + c^2) + \ldots + \ldots = ab^2 + abc + ac^2 + bc^2$

$$+ abc + ba^2 + ca^2 + abc + b^2c$$

$$= (a + b + c)(bc + ca + ab) = 0.$$

(ii.) Prove that

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

Call the left-hand expression A. Then $A=0$ when $y=z$, *i.e.* when $y-z=0$.

$\therefore$ A is divisible by $y-z$; similarly, A is divisible by $z-x$ and by $x-y$. [See § 2, p. 112.] $\therefore$ $A=B(y-z)(z-x)(x-y)$.

B cannot contain x, y, or z, for the right-hand member is already of the third degree. $\therefore$ B must be simply a *number*.

Put $x=0$, $y=1$, $z=2$, then $-1+8-1=B(-1)\times 2\times(-1)$,

$$\therefore 6=2B, \quad \therefore B=3, \text{ and } \quad \therefore A=3(y-z)(z-x)(x-y).$$

(iii.) Prove that $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4$
$+a^4+b^4+c^4=12abc\,(a+b+c).$

If we put $a=0$, $b=0$, $c=0$ in succession, the left-hand member vanishes, $\therefore$ it$=Mabc$. Now, $\because$ it is of the fourth degree and *symmetrical* with respect to a, b, c, $\therefore$ M must be a *symmetrical* expression containing a, b, c in the first degree only, $\therefore$ $M=N\,(a+b+c)$ where N is necessarily a *number*. Thus $(a+b+c)^4 \ldots =Nabc(a+b+c)$. To determine N, put $a=b=c=1$; then $81-3\times 16+3=N.3$, $\therefore$ $3N=36$, $\therefore$ $N=12$,

$$\therefore \text{the given expression}=12abc(a+b+c).$$

(iv.) In how many different ways can £5 be made up of sovereigns, half-crowns, and shillings, one coin at least of each kind being used?

(α.) 1 sovereign, and 2, 4, 6 . . . or 30 half-crowns, and the remainder in shillings: which gives 15 ways.

(β.) 2 sovereigns, and 2, 4 . . . or 22 half-crowns: 11 ways.

(γ.) 3 sovereigns, and 2, 4 . . . or 14 half-crowns: 7 ways.

(δ.) 4 sovereigns, and 2, 4, or 6 half-crowns: 3 ways.

Therefore the number of ways $=36$.

(v.) Find the term in the product $(1+x-x^2)^2\times(1-x+x^2)^2$ which involves x^2.

$$(1+x-x^2)^2\times(1-x+x^2)^2=[\{1+(x-x^2)\}\{1-(x-x^2)\}]^2=\{1-(x-x^2)^2\}^2$$
$$=(1-x^2+2x^3-x^4)^2=\ldots-2x^2+\ldots \quad \text{[See § 2, p. 1.]}$$

Therefore $-2x^2$ is the required term.

(vi.) Eliminate x from $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$.

$$\left.\begin{aligned} aa'x^2+a'bx+ca'&=0 \\ \text{and } aa'x^2+ab'x+c'a&=0 \end{aligned}\right\} \qquad \therefore (a'b-ab')x+ca'-c'a=0, \quad \therefore x=-\frac{ca'-c'a}{a'b-ab'}.$$

$$\text{Again, } \left.\begin{aligned} c'ax^2+bc'x+cc'&=0 \\ \text{and } ca'x^2+b'cx+cc'&=0 \end{aligned}\right\} \qquad \therefore (c'a-ca')x^2+(bc'-b'c)x=0, \quad \therefore x=-\frac{bc'-b'c}{c'a-ca'},$$

Equating the two values of x, we get $\dfrac{ca'-c'a}{a'b-ab'}=\dfrac{bc'-b'c}{c'a-ca'}$,

$$\therefore (ca'-c'a)^2=(ab'-a'b)(bc'-b'c).$$

(vii.) If $A+Bx+Cx^2=a+bx+cx^2$ for three different values of x, then $A=a$, $B=b$, $C=c$.

Let the three values be α, β, γ.

$$\therefore \left.\begin{aligned} A+B\alpha+C\alpha^2&=a+b\alpha+c\alpha^2 \\ A+B\beta+C\beta^2&=a+b\beta+c\beta^2 \\ A+B\gamma+C\gamma^2&=a+b\gamma+c\gamma^2 \end{aligned}\right\}.$$

$$\therefore \left.\begin{aligned} B(\alpha-\beta)+C(\alpha^2-\beta^2)&=b(\alpha-\beta)+c(\alpha^2-\beta^2) \\ B(\beta-\gamma)+C(\beta^2-\gamma^2)&=b(\beta-\gamma)+c(\beta^2-\gamma^2) \end{aligned}\right\} \text{ by subtraction,}$$

$$\therefore \left.\begin{aligned} B+C(\alpha+\beta)&=b+c(\alpha+\beta) \\ B+C(\beta+\gamma)&=b+c(\beta+\gamma) \end{aligned}\right\} \text{ by division by } \alpha-\beta \text{ and } \beta-\gamma.$$

$\therefore$ by subtraction $C(\alpha-\gamma)=c(\alpha-\gamma)$, $\therefore C=c$ by division by $\alpha-\gamma$,

and $\therefore$ also $A=a$, $B=b$. [See § 4, p. 20.]

(viii.) Between what values must x lie in order that $2x^2+5x-3$ may be negative.

$$2x^2+5x-3=(2x-1)(x+3)=2(x-\tfrac{1}{2})(x+3).$$

Therefore if $x>+\frac{1}{2}$, both factors are positive,

if $x<-3$, both factors are negative,

but if x is $<+\frac{1}{2}$ and >-3, $x-\frac{1}{2}$ is negative and $x+3$ is positive.

Therefore in order that $2x^2+5x-3$ may be negative, x must lie between $+\frac{1}{2}$ and -3.

(ix.) Find the value of $\dfrac{2x^3-7x^2+12}{x^3-7x+6}$ when $x=2$.

If we put $x=2$, we get $\frac{16-28+12}{8-14+6}=\frac{0}{0}$, which is *indeterminate;* but since both numerator and denominator $=0$ when $x=2$,

$$\therefore\ x-2 \text{ is a factor.}$$

Dividing out we get $\frac{(x-2)(2x^2-3x-6)}{(x-2)(x^2+2x-3)}=\frac{2x^2-3x-6}{x^2+2x-3}$.

Now put $x=2$: thus we get $\frac{8-6-6}{4+4-3}=-\frac{4}{5}$.

(x.) Eliminate x from

$$yx^2-5x+4y=0, \text{ and } (y-2)x^2-2x+5y-2=0.$$

$$\left.\begin{aligned} y(y-2)x^2-(5y-10)x+4y^2-8y&=0\\ y(y-2)x^2-2yx+5y^2-2y\qquad&=0\end{aligned}\right\}$$

$$\therefore\ (3y-10)x=-y^2-6y\quad \therefore\ x=\frac{-y^2-6y}{3y-10},$$

$$\therefore\ y\left(\frac{y^2+6y}{3y-10}\right)^2+\frac{5y^2+30y}{3y-10}+4y=0$$

$$\therefore\ y^4+12y^3+87y^2-200y+100=0.$$

(xi.) Given $x-\sqrt{x-1}+\sqrt[3]{x+1}=0$, obtain an equation of the sixth degree in x free of roots.

$$(x-\sqrt{x-1})^3=(-\sqrt[3]{x+1})^3,$$

$$\therefore\ x^3-3x^2\sqrt{x-1}+3x(x-1)-(x-1)\sqrt{x-1}=-x-1,$$

$$\therefore\ -(3x^2+x-1)\sqrt{x-1}=-x^3-3x^2+2x-1,$$

$$\therefore\ (9x^4+x^2+1+6x^3-6x^2-2x)(x-1)=x^6+9x^4+4x^2+1+6x^5-4x^4$$
$$+2x^3-12x^3+6x^2-4x,$$

$$\therefore\ x^6-3x^5+8x^4+x^3+7x^2-7x+2=0.$$

(xii.) Eliminate x, y from the equations

$$x+y=a,\ x^2+y^2=b^2,\ x^3+y^3=c^3.$$

$$(x+y)^2-(x^2+y^2)=a^2-b^2,\quad \therefore\ xy=\frac{a^2-b^2}{2}.$$

$$x^3+y^3=(x+y)(x^2-xy+y^2)=a\left(b^2-\frac{a^2-b^2}{2}\right)=\frac{a(3b^2-a^2)}{2},$$

$$\therefore\ c^3=\frac{a(3b^2-a^2)}{2},\quad \therefore\ a(3b^2-a^2)-2c^3=0.$$

(xiii.) Determine values of p, q which will make $4y^4-12y^3+py^2+qy+16$ a perfect square for all values of y.

$$\begin{array}{rl}
 & 4y^4-12y^3+py^2+qy+16(2y^2-3y+4 \\
 & 4y^4 \\
\hline
4y^2-3y) & -12y^3+py^2 \\
 & -12y^3+9y^2 \\
\hline
4y^2-6y+4) & (p-9)y^2+qy+16 \\
 & 16y^2-24y+16 \\
\hline
 & (p-25)y^2+(q+24)y.
\end{array}$$

If this remainder$=0$, the given expression is a perfect square,

$$\therefore p=25 \text{ and } q=-24.$$

Otherwise, assume $4y^4-12y^3+py^2+qy+16=(2y^2+ay+4)^2=4y^4+a^2y^2+16+4ay^3+16y^2+8ay$. Equating co-efficients, we get $a=-3$, $q=-24$, and $p=25$.

(xiv.) If x be real, show that $x^2-8x+20$ can never be<4.

$$x^2-8x+20=(x^2-8x+16)+4=(x-4)^2+4.$$

But $\because$ x is real, $\therefore$ $(x-4)^2$ must be either 0 or>0.

Therefore $x^2-8x+20$ cannot be<4.

Otherwise, assume $x^2-8x+20=y$: then $x=4\pm\sqrt{y-4}$ $\therefore$ y cannot be<4.

(xv.) Find the first term of the third order of differences of the series 1, 5, 15, 35, 70.

The rule for finding the several orders of differences of a given series of terms is the following :—

(i.) Take the first term from the second, the second from the third, the third from the fourth, and so on ; the remainders will form the *first order of differences.*

(ii.) Take the first term of this last series from the second, the second from the third, and so on ; the remainders will form the *second order of differences.*

(iii.) Proceed in the same manner for the third, fourth, etc., orders of differences.

Thus we get in the example given—

$$\begin{array}{ccccccccc}
1 & & 5 & & 15 & & 35 & & 70 \\
 & 4 & & 10 & & 20 & & 35 & \\
 & & 6 & & 10 & & 15 & & \\
 & & & 4 & & 5 & & &
\end{array}$$

$$\therefore \text{required term} = 4.$$

EXAMPLES. XVI.

1. Find the co-efficients of x^2, x^3 and x^4 in $(x+a)^3 \times (x-a)^5$.

2. Prove $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=(y+z)(z+x)(x+y)$.

3. Prove that $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2-12abc$ is divisible by $a+b+c$, and find the other factor.

4. Simplify $\left\{\frac{\sqrt{x+a}}{\sqrt{x-a}}-\frac{\sqrt{x-a}}{\sqrt{x+a}}\right\} \times \frac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}$.

5. Find the four factors of $(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4$.

6. Find the limits between which $\frac{x^2-3x-3}{2x^2+2x+1}$ will lie for all real values of x.

7. Show that $(a-x)(x+\sqrt{x^2+b^2})$ cannot exceed $\frac{a^2+b^2}{2}$ for possible values of x.

8. If x be a real quantity, prove that $\frac{2x^2+6x+3}{2x+1}$ must be either <1 or >3.

9. If $1+a+a^2=0$, prove that $a^3=1$, and that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+ay+a^2z)(x+a^2y+az).$$

10. If a be a large integer, and N an integer between a^3 and $(a+1)^3$ prove that $\sqrt[3]{N}=\frac{2N+a^3}{2a^3+N}\cdot a$, nearly.

11. Two men A and B set out at the same time in the same direction. A travels 8 miles a day; B travels the first day 1 mile, the second day 2 miles, the third day 3 miles, and so on. In how many days will B overtake A?

12. Find the value of the following Indeterminate Forms:—

(i.) $\left(\frac{x^3-a^3}{x^2-a^2}\right)^2$, when $x=a$.

(ii.) $\frac{x^3-1}{x^3-2x^2+2x-1}$, when $x=1$.

(iii.) $\dfrac{1+x-x^2-x^3}{1+2x+2x^2+2x^3+x^4}$, when $x=-1$.

(iv.) $\dfrac{x^3-5x^2+8x-6}{x^3-7x^2+16x-12}$, when $x=3$.

(v.) $\dfrac{x\epsilon^{2x}+1-\epsilon^{2x}-x}{\epsilon^{2x}-1}$, when $x=0$.

(vi.) $\dfrac{2}{x^2-1}-\dfrac{1}{x-1}$, when $x=1$.

(vii.) $\dfrac{a(x^2+c^2)-2acx}{b(x^2+c^2)-2bcx}$, when $x=c$.

(viii.) $\dfrac{x^3-19x+30}{x^3-2x^2-9x+18}$, when $x=2$ and when $x=3$.

13. If $p_r=\dfrac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}$, prove that

$$p_n+p_{n-1}p_1+p_{2-2}p_2+\ldots+p_1p_{n-1}+p_n=1.$$

14. Prove that $(x+y+z)^3-x^3-y^3-z^3=3(y+z)(z+x)(x+y)$.

15. Prove that $(x+y+z)^5-x^5-y^5-z^5$ is divisible by $(y+z)(z+x)(x+y)$, and find the other factor.

16. Prove that $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0$ is equivalent to the equation $(ax+by-1)^2+(bx-ay)^2=0$; and hence that the only possible values of x, y are $\dfrac{a}{a^2+b^2}, \dfrac{b}{a^2+b^2}$.

17. If $x=\dfrac{ab-cd}{(a-b)-(c-d)}$ prove that $(x+a)(x-b)=(x+c)(x-d)$, and that $\dfrac{x+a}{x-b}=\dfrac{(a-c)(a+d)}{(b-d)(b+c)}$.

18. If $2s=a+b+c$ and $2\sigma^3=a^3+b^3+c^3$, prove that

$$(\sigma^3-a^3)(s-a)+(\sigma^3-b^3)(s-b)+(\sigma^3-c^3)(s-c)=a^4+b^4+c^4-s\sigma^3.$$

19. If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

20. Prove that $4(a^2+ab+b^2)^3-27a^2b^2(a+b)^2=(a-b)^2(a+2b)^2(2a+b)^2$.

21. Find the factors of $x^3-9x^2+11x+21$.

22. If $x-\dfrac{1}{x}=1$, prove that $x^2+\dfrac{1}{x^2}=3$ and $x^3-\dfrac{1}{x^3}=4$.

23. Prove that
$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+3abc=(a+b+c)(bc+ca+ab).$$

24. If ax^2+bx+c becomes 8, 22, 42 respectively when $x=2$, 3, 4: find its value when $x=-\frac{1}{3}$.

25. Eliminate x between the equations $x^3-px^2+qx-r=0$ and $x=\sqrt{y}+1$, expressing the resulting equation free of surds and arranged according to descending powers of y.

26. If $ab+bc+ca=0$, prove that $\dfrac{1}{a^2-bc}+\dfrac{1}{b^2-ca}+\dfrac{1}{c^2-ab}=0$.

27. If $ax^4+bx^3+cx^2+dx+e$ is a perfect square, prove that $\dfrac{a}{e}=\dfrac{b^2}{d^2}$.

28. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, prove that $\dfrac{x}{a}=\dfrac{y}{b}$.

29. Write down the square of $\dfrac{a}{3}-\dfrac{b^2}{5}+\dfrac{c^3}{7}-\dfrac{d^4}{9}$.

30. Write down the quotient of x^2-y^4 divided by $x^{\frac{1}{3}}+y^{\frac{2}{3}}$.

31. There are two numbers whose product is 192, and the quotient of the arithmetical mean by the harmonical mean of their G. C. M. and L. C. M. $=3\frac{25}{48}$: find the numbers.

32. If in a given time A can do $\left(\dfrac{1}{p}\right)^{th}$ part of the work that B and C can do together, B $\left(\dfrac{1}{q}\right)^{th}$ part of what C and A can do together, and C $\left(\dfrac{1}{r}\right)^{th}$ part of what A and B can do together; compare the times that each would require to perform separately a fixed amount of work.

33. If $a+b+c=0$, prove that $a^2-bc=b^2-ca=c^2-ab$.

34. Find the maximum value of $\dfrac{x^2+x+1}{x^2-x+1}$ for real values of x.

35. Split $\dfrac{3x-2}{(x+2)(x+1)x}$ into partial fractions. And so sum the series $\dfrac{1}{1.2.3}+\dfrac{4}{2.3.4}+\dfrac{7}{3.4.5}+\dfrac{10}{4.5.6}+\ldots$ to ∞.

36. Show that the value of $(a^2-bc)+(b^2-ca)+(c^2-ab)$ is not altered if a, b, c are each of them increased or diminished by the same quantity.

37. Show that $(a+b+c)(ax^2+by^2+cz^2)=(ax+by+cz)^2+bc(y-z)^2$
$+ca(z-x)^2+ab(x-y)^2$.

38. Multiply $\frac{1}{3}x^{\frac{1}{2}}-\frac{1}{2}y^3$ by $\frac{1}{9}x+\frac{1}{6}x^{\frac{1}{2}}y^3+\frac{1}{4}y^6$.

39. Show that $\dfrac{x^2-3x+4}{x^2+3x+4}$ must lie between 7 and $\frac{1}{7}$ for real values of x.

40. Simplify $\dfrac{\frac{1}{a^2}-\left(\frac{1}{b}-\frac{1}{c}\right)^2}{\frac{1}{b^2}-\left(\frac{1}{a}+\frac{1}{c}\right)^2}+\dfrac{\frac{1}{b^2}-\left(\frac{1}{c}-\frac{1}{a}\right)^2}{\frac{1}{c^2}-\left(\frac{1}{b}+\frac{1}{a}\right)^2}+\dfrac{\frac{1}{c^2}-\left(\frac{1}{a}-\frac{1}{b}\right)^2}{\frac{1}{a^2}-\left(\frac{1}{c}+\frac{1}{b}\right)^2}$.

41. If $x=a+\dfrac{1}{a}$ and $y=b+\dfrac{1}{b}$, prove that

$$\sqrt{(xy+2x+2y+4)}=(a^{\frac{1}{2}}+a^{-\frac{1}{2}})(b^{\frac{1}{2}}+b^{-\frac{1}{2}}).$$

42. Find the minimum value of x^2-2x+3 for possible values of x.

43. If ax^3+bx^2+cx+d is a perfect cube, prove that $\dfrac{a}{d}=\dfrac{b^3}{c^3}$.

44. Divide

$$a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3) \text{ by } a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2).$$

45. Simplify $\dfrac{\left(\dfrac{a+b-c}{c^2-a^2+2ab-b^2}\right)^{\frac{1}{2}}+\left(\dfrac{a+c-b}{b^2-a^2+2ac-c^2}\right)^{\frac{1}{2}}}{\left(\dfrac{a+c-b}{b^2-a^2+2bc+c^2}\right)^{\frac{1}{2}}+\left(\dfrac{b+c-a}{a^2-b^2+2ac+c^2}\right)^{\frac{1}{2}}}$.

46. To complete a piece of work A takes twice as long as B and C together, and B takes thrice as long as A and C together; compare the time that C would take with that which B and C together would take.

47. Prove that $\{(x-y)^2+3xy\}\{(x+y)^2-3xy\}=(x^2-y^2)^2+3x^2y^2$; and find the factors of

$$abx^2-(a^2+b^2)xy+aby^2,\ b^3+c^3-3bc+1, \text{ and } x^4+y^4-2(x^2+y^2)+1.$$

48. A and B set out together from the same place. A goes 8 miles the first day, 12 the second, 16 the third, and so on; B goes 1 mile the first day, 4 the second, 9 the third, and so on. How many days will they travel before B overtakes A?

49. Express $x^2+20y^2+70z^2+52yz+4xy+6zx$ as the sum of three squares.

50. Find the square root of $x+y+\sqrt{x^2-y^2}+\sqrt{2xy+2y^2}+\sqrt{2xy-2y^2}$.

51. Show that if n be an even integer

$$\frac{1}{\underline{|1}\,\underline{|n-1}}+\frac{1}{\underline{|3}\,\underline{|n-3}}+\frac{1}{\underline{|5}\,\underline{|n-5}}+\ldots+\frac{1}{\underline{|n-1}\,\underline{|1}}=\frac{2^{n-1}}{\underline{|n}}.$$

52. Prove that the squares of x^2-4x+2, x^2-2x+2, and x^2-2 are in A. P.

53. Prove that

$$1+\frac{3}{2^3}+\frac{1.3}{1.2}\cdot\frac{3^2}{2^6}+\frac{1.3.5}{1.2.3}\cdot\frac{3^3}{2^9}+\ldots \text{ to } \infty \;=2.$$

54. The sum of three numbers in H. P. is 13, and the sum of their squares is 61. Find them.

55. Find the cube root of 1003 to five places of decimals by the Binomial Theorem.

56. If the arithmetic mean between two numbers is 1, prove that the harmonic mean is the square of the geometric mean.

57. Extract the square root of

$$81(x^4+1)+36x(x^2-1)-158x^2.$$

58. If $a+b+c+d=0$, prove that

$$abc+bcd+cda+dab=\sqrt{(bc-ad)(ca-bd)(ab-cd)}.$$

59. Express $(b^2-c^2)(1+ab)(1+ac)$
$+(c^2-a^2)(1+bc)(1+ba)$
$+(a^2-b^2)(1+ca)(1+cb)$ as the product of four factors.

60. Add together $\dfrac{2}{5-3x}$, $\dfrac{3x+4}{(5-3x)^2}$, and $\dfrac{5x+6x^2+7}{(5-3x)^3}$:

and take $\dfrac{1}{(1-3x-5x^2)^3}$ from $\dfrac{x^2-3x+1}{(1-3x-5x^2)^4}$.

61. Multiply $6x-7+\dfrac{33x-18x^2-13}{3x-2}$ by $4x+9-\dfrac{2x^2+32x-27}{5x-3}$.

62. Find the greatest common measure of

$$x^4-9x^3+29x^2-39x+18 \text{ and } 4x^3-27x^2+58x-39.$$

63. Solve the following equations:

(i.) $\sqrt{(x+3)}\times\sqrt{(3x-3)}=24$. (ii.) $\sqrt{(x+2)}+\sqrt{(3x+4)}=8$.

(iii.) $3x^3+\frac{13}{3}x^3-39x=81$.

64. Find a mean proportional to 441 and 841; and a third proportional to 81 and 99.

65. When are the clock hands at right angles first after twelve o'clock?

66. If the diameter of a guinea is to that of a sovereign as 9 : 7, in what ratio should their thicknesses be?

67. A number divided by the product of its digits gives as quotient 2, and the digits are inverted by adding 27. What number is it?

68. A bill of £26, 15s. was paid with half-guineas and crowns, and the number of half-guineas exceeded the number of crowns by 17. How many were there of each?

69. Prove that

$$(ac+1)^2(bc+1)^2-\{(ac-1)(bc-1)+2c\}^2=4c(a+b-1)(abc^2+c+1),$$

and show that the expression $\frac{x^2-4x+4}{x-1}$ cannot lie between 0 and -4.

70. Solve the equations

$$\left.\begin{aligned} x^2y+xy^2&=ab \\ x^3+y^3&=a^2-ab+b^2 \end{aligned}\right\}.$$

71. If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, form the equation of which the roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

72. From a given point in a circular race-course, A starts to walk two miles against B running three, and loses by a quarter of the length of the course. It is observed that A is not passed by B until the former has walked $1\frac{2}{3}$ of a mile. Find the circumference of the course.

73. Find the number of spots upon a set of dominoes ranging from double blank to double n: and prove that it is a multiple of 3.

74. Find the greatest common measure of

$$x^4-15x^3+81x^2-185x+150, \text{ and } 4x^3-45x^2+162x-185.$$

75. Add $\frac{1}{2+3x}, \frac{2x-5}{(2+3x)^2}, \frac{x^2-6x-7}{(2+3x)^3}$. And take $\frac{1}{1+x+x^2}$ from $\frac{1}{1-x-x^2}$.

76. Multiply $3+5x-\frac{12+41x+36x^2}{4+7x}$ by $5-2x+\frac{26x-8x^2-14}{3-4x}$.

77. Solve the following equations:

(i.) $\sqrt{\dfrac{135}{x+1}+1}=4.$ (ii.) $\sqrt{(x+3)}+\sqrt{(3x-3)}=10.$

(iii.) $x+y=6$; $(x^2+y^2)(x^3+y^3)=1440.$

78. Find a mean proportional to $2\frac{1}{2}$ and $5\frac{5}{8}$, and a third proportional to 100 and 130.

79. Find two numbers in the proportion of 9 to 7 such that the square of their sum shall be equal to the cube of their difference.

80. The ten's digit of a number is less by 2 than the unit's digit, and if the digits are inverted the new number is to the former as 7 : 4. Find the digits.

81. Forty sheep are bought for £100, 20 are sold at 10 per cent. profit. At what price per head must the rest be sold to clear 20 per cent. on the whole lot?

82. Prove that

$$(1+x)^n+n(1+x)^{n-1}x+\frac{n(n+1)}{\underline{|2}}(1+x)^{n-2}x^2+\ldots\text{to }\infty=(1+x)^{2n}.$$

83. Simplify

(i.) $(x-a)(x-b)(x-c)-[bc(x-a)-x\{(a+b+c)x-a(b+c)\}]$:

and (ii.) $$\frac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}}\div\frac{1-\dfrac{a^2-b^2}{a^2+b^2}}{1+\dfrac{a^2-b^2}{a^2+b^2}}.$$

84. Extract the sixth root of

$$a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20.$$

85. Show that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\sqrt{2}.$

86. For what values of x will $2x^2+x-6$ be *positive*?

87. A tradesman sells his goods at a price which gives a profit of x per cent. on the selling price, and $x+50$ per cent. on the cost price. If his profit for a single day be £20, find his gross receipts for that day.

88. If the roots of $x^2+px+q=0$ and $x^2+qx+p=0$ differ by the same quantity, and p be not equal to q, prove that $p+q+4=0$.

89. A man writes three letters and directs three envelopes into which he puts them at random: find the chance that all the letters will go wrong.

90. Find the co-efficient of x^6 in the expansion of $\dfrac{7x+6}{(3-x)(2x+1)}$.

91. Prove that $(a^{n-1}-b^{n-1})(a^n-b^n)(a^{n+1}-b^{n+1})$ is divisible by $(a-b)(a^2-b^2)(a^3-b^3)$.

92. Find the L. C. M. of $x^3-4x^2+7x-12$, $2x^3-3x^2+9x-4$, and $4x^3-16x^2+13x-3$.

93. If any two of the fractions $\dfrac{(a+b)(c+d)}{ab+cd}$, $\dfrac{(a+c)(b+d)}{ac+bd}$, and $\dfrac{(a+d)(b+c)}{ad+bc}$ be equal to one another, then each must be equal to the third and also to -1; a, b, c being unequal quantities.

94. Prove that $\dfrac{7+3\sqrt{5}}{\sqrt{2}+\sqrt{7+3\sqrt{5}}}+\dfrac{7-3\sqrt{5}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}=2\sqrt{2}$: and determine which is the greater, $\sqrt{10}+\sqrt{7}$ or $\sqrt{19}+\sqrt{3}$.

95. Solve the equations:

$$\text{(i.)}\ \frac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$$

$$\text{and (ii.)}\ \left(\frac{x+2a}{x+2b}\right)^{\frac{1}{2}}=\frac{x+a}{x+b}.$$

96. Supposing that each of six courts sends an ambassador to each of the others, how many ambassadors would there be, and in how many different ways could they be distributed?

97. A pound of tea and three pounds of sugar cost six shillings; but if sugar were to rise 50 per cent., and tea 10 per cent., they would cost seven shillings. What are the prices of tea and sugar?

98. Find the co-efficient of x in the product $(x+2)(x-6)(x-10)(x+14)$.

99. The product of two factors is $(2x+3y)^3+(2y+3z)^3$, and one of the factors is $2x+5y+3z$: find the other.

100. Reduce $\dfrac{10x^4-7x^3+x^2}{4x^4-2x^3-2x+1}$ to its lowest terms, and simplify

$$\text{(i.)}\ \frac{a+b-c}{(b-c)(c-a)}+\frac{b+c-a}{(c-a)(a-b)}+\frac{(c+a-b)}{(a-b)(b-c)}.$$

$$\text{(ii.)}\ \left\{\frac{x+2y}{x+y}+\frac{x}{y}\right\}\div\left\{\frac{y}{x+y}+\frac{x+y}{y}\right\}.$$

101. If $a+\frac{1}{b}=1$, and $c+\frac{1}{a}=1$, prove that $b+\frac{1}{c}=1$.

102. Find the value of $\frac{x^2+xy+y^2}{x^2-xy+y^2}$ when $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

103. Solve the equations :

$$\text{(i.) } \frac{1}{2}\left\{\frac{2x}{3}+4\right\}-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left\{\frac{6}{x}-1\right\}.$$

$$\text{(ii.) } (x+2a)(x-a)^2=(x+2b)(x-b)^2.$$

$$\text{(iii.) } \frac{a-(a^2-x^2)^{\frac{1}{2}}}{a+(a^2-x^2)^{\frac{1}{2}}}=b.$$

104. Express in factors

$$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz,$$

and if $A=a^2-bc$, $B=b^2-ac$, $C=c^2-ab$: prove that

$$(A^2-BC)bc=(B^2-AC)ac=(C^2-AB)ab=abc(a+b+c)(A+B+C).$$

105. Reduce to its simplest form the fraction

$$\left\{\frac{a^2+b^2-c^2}{ab}-\frac{b^2+c^2-a^2}{bc}\right\}\left\{\frac{a+b+c}{a+c-b}-\frac{a+c-b}{a+b+c}\right\}.$$

106. A and B gained by trading £100. Half of A's stock was less than B's by £100, and A's gain was $\frac{3}{20}$ of B's stock : what did each put into the stock, and what are their respective shares of the gain ?

107. A number consisting of three digits is equal to 42 times the sum of the middle and left-land digits, the sum of the digits is equal to 9, and the right-hand digit is twice the sum of the other two : what is the number ?

108. Solve the equations :

$$\text{(i.) } \left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z}&=7\tfrac{3}{5} \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}&=10\tfrac{1}{6} \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}&=16\tfrac{1}{10} \end{aligned}\right\}.$$

$$\text{(ii.) } \frac{x-7}{x-5}-\frac{x-8}{x-6}=\frac{2x-7}{2x-5}-\frac{2x-11}{2x-9}.$$

109. A and B distribute £5 each in charity: A relieves 5 persons more than B, but B gives to each 1s. more than A. How many did they each relieve?

110. Add together $\sqrt{12}$, $3\sqrt{75}$, $\frac{1}{2}\sqrt{147}$, and $\frac{2}{3}\sqrt{\frac{4}{75}}$.

$$\text{Simplify } \frac{3+2\sqrt{2}}{3-2\sqrt{2}} \text{ and } \sqrt{38-12\sqrt{10}}.$$

111. Find the sum of 8 terms of the following series:—

$$\text{(i.) } 3\tfrac{3}{8}+2\tfrac{1}{4}+1\tfrac{1}{2}+1+\ \ldots$$

$$\text{(ii.) } 3\tfrac{3}{8}+2\tfrac{1}{4}+1\tfrac{1}{8}+0-\ \ldots$$

112. The number of combinations of $n+1$ things 4 together, is 9 times the number of combinations of n things 2 together; find n.

113. Expand $(1+2x)^{\frac{1}{2}}$ to five terms by the Binomial Theorem, and show that the result is the same as that obtained by extracting the square root of $1+2x$ by the usual method.

114. If the scale of relation of a recurring series

$$a+bx+cx^2+\ \ldots\ +kx^{n-2}+lx^{n-1} \text{ be } 1-rx-sx^2,$$

find the sum of the first n terms.

115. Solve the equations:

$$\text{(i.) } \left.\begin{array}{r} x^2y+xy^2=120 \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{8}{15} \end{array}\right\}.$$

$$\text{(ii.) } \left.\begin{array}{r} 3xy-y^2=35 \\ 3y^2-x^2=59 \end{array}\right\}.$$

116. For a journey of 108 miles 6 hours less than would have sufficed had one gone 3 miles an hour faster. How many miles an hour did one go?

117. Simplify the fractions:

$$\text{(i.) } \frac{x}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^3}{(1-x)^3}.$$

$$\text{(ii.) } \frac{x^4-a^4}{x^2-2ax+a^2}\times\frac{x-a}{x^2+ax}\times\frac{x^5-a^2x^3}{x^3+a^3}\times\frac{x^2-ax+a^2}{x^4-2ax^3+a^2x^2}.$$

118. Find the G. C. M. of $4x^4-4x^2y^2+4xy^3-y^4$, and

$$6x^4+4x^3y-9x^2y^2-3xy^3+2y^4.$$

Hence find also the L. C. M. of these two quantities.

119. Divide 3756025 by 6 in the scale of 8.
Express 215855 in the scale of 12, and 23·125 in the scale of 9.

120. Find the eighth term in the expansion of $(1+x)^{11}$, and the r^{th} term in that of $(1-2x)^{-\frac{1}{2}}$: and expand $(1-x^2)^{-\frac{3}{2}}$ to five terms.

121. If the Geometrical Mean between two quantities x and y be to the Harmonical Mean as $1 : n$, prove that x is to y as $1+\sqrt{1-n^2}$ is to $1-\sqrt{1-n^2}$.

Hence show that the Geometrical Mean between two real quantities is greater than the Harmonical Mean.

122. A purse contains a certain number of sovereigns, three times as many shillings, and five times as many pence, and the whole sum is £281; find how many sovereigns, shillings, and pence it contains.

123. If P and Q be the nth terms in the expansion of

$$(a^2-x^2)^{-\frac{1}{2}} \text{ and } (a^2-x^2)^{-\frac{3}{2}}, \text{ prove that } \frac{P}{Q}=\frac{a^2}{2n-1}.$$

124. A sum of money is distributed amongst a certain number of persons. The second receives 1s. more than the first, the third 2s. more than the second, the fourth 3s. more than the third, and so on. If the first person receives 1s., and the last £3, 7s., find the number of persons and the sum distributed.

125. Find the chance of throwing 6 with two dice: and the odds against throwing 6 twice at least in three throws with two dice.

126. Write down the last three terms of the quotient of $x+y$ divided by $x^{\frac{1}{2n+1}}+y^{\frac{1}{2n+1}}$, n being a positive integer. How many terms will there be in the quotient?

127. Solve the equations:

(i.) $2cx^2-abx+2abd=4cdx.$

(ii.) $(x+y)^2-z^2=65,\ x^2-(y+z)^2=13,\ x+z-y=9.$

128. A bill of £63, 5s. was paid in sovereigns and half-crowns, and the number of coins used in the payment was 100: how many of each kind were used?

129. Prove that

$$\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-c)(b-a)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

130. Find the square root of

$$\frac{x^4}{4}+ax^3+\frac{4a^2x^2}{3}+\frac{2a^3x}{3}+\frac{a^4}{9}.$$

131. Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$, when $x=\dfrac{4ab}{a+b}$.

132. Divide $a-b^2$ by $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$.

133. A cubical tank contains 512 cubic feet of water. It was required to enlarge the tank, the depth remaining the same, so that it should contain 7 times as much water as before, subject to the condition that the length which is added to one side of the base should be 4 times the length added to the other side. Find the sides of the new rectangular base.

134. A watch which is set right at noon gains two minutes the first hour afterwards, three the second, four the third, and so on: after what interval will the watch be an hour and a half fast, and what time will it then indicate?

135. If the series which are the expansions of $(1-x)^n$ and $(1-x)^{-n}$ be multiplied together, find the co-efficient of x^4 in the product.

136. Find a G. P. such that the sum of the first two terms may be 12 and of the first three may be 39.

137. Find the square root of

$$3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2.$$

138. Add a term to $2px^2+3qx$ which shall make the result a perfect square for all values of x.

139. Simplify (i.) $\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)\cdot\dfrac{1}{y^2-x^2}-\dfrac{y}{x^2+xy}+\dfrac{x}{xy-y^2}$:

and (ii.) $x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2\div\dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

140. If α, β are the roots of $lx^2-2mx+n=0$, prove that $\dfrac{1}{\alpha}+\dfrac{1}{\beta}-\dfrac{2m}{n}=0$.

141. Find the first four terms of $(y-y^3+y^5-y^7+\ldots)^2$.

142. Assuming that $\dfrac{c^2}{c^2+2cy-y^2}=A+By+Cy^2+Dy^3+\ldots$ find the values of A, B, C, D.

143. If $2(x^2+a^2-ax)(y^2+b^2-by)=x^2y^2+a^2b^2$, prove that

$$(x-a)^2(y-b)^2+(bx-ay)^2=0,$$

and hence find the only possible values of x, y.

144. If $\dfrac{n^4+n^2-1}{n^3+n^2+n+1}$ be converted into a continued fraction, prove that the quotients will be $n-1$ and $n+1$ alternately: and find the successive convergents.

145. If $y=2x$ and $a=3b$, simplify

$$\frac{(ax+by)(ay+bx)(a^2+x^2)}{(bx-ay)(by-ax)(b^2+y^2)}.$$

146. Find the number of arrangements of the expanded form of the expression $a^m b^2$ such that the two b's come together in each arrangement; and prove that the number of arrangements in which the two b's never come together is equal to $\dfrac{m(m+1)}{2}$.

147. If α, β are the roots of $x^2-px+q=0$, form the equation whose roots are $\dfrac{\alpha}{\beta^2}$ and $\dfrac{\beta}{\alpha^2}$.

148. Find the sum of $2n+1$ terms of the series $a+(a+d)+(a+2d)+\ldots$ beginning with the $(n+1)^{th}$ term.

149. Simplify $\sqrt[3]{a^6 b^{\frac{3}{2}} c^{\frac{3}{4}}} \times b^{\frac{1}{2}} \times (c^{\frac{1}{4}} a^2)^{-\frac{1}{4}}$.

150. Of 12 men, 2 can steer and cannot row, and the rest can row but cannot steer: in how many ways can the crew of an eight-oar, with a coxswain, be made up?

151. Solve the equations:

$$\text{(i.)}\quad \frac{3x-2}{4}+\frac{x}{2}-11\tfrac{5}{6}=\frac{x-\dfrac{4x-9}{3}}{6}-5.$$

$$\text{(ii.)}\quad (x^2-bx+b^2)(ax+ab+b^2)=ax^3+a^2b^2+b^4.$$

152. Find the *scale of relation* in the recurring series $2+3x+19x^2+101x^3+\ldots$ having given that it is a quadratic expression.

153. Prove that

$$(1+x+x^2+x^3+\ldots)^4=1+4x+10x^2+20x^3+35x^4+\ldots$$

154. Find the sum of

$$\frac{n-1}{nx}+\frac{n-2}{nx^2}+\frac{n-3}{nx^3}+\ldots \text{ to } n \text{ terms.}$$

155. A man walking from A to B, at the rate of 4 miles an hour, starts one hour before a coach travelling 12 miles an hour, and is picked up by the coach. On reaching B he finds that his coach journey has lasted two hours. Find the distance from A to B.

156. Prove that $\left(x+\frac{1}{x}\right)^6-\left(x-\frac{1}{x}\right)^6=12\left(x^4+\frac{1}{x^4}+\frac{10}{3}\right)$.

157. If the second term of an A. P. be a geometric mean between the first and fourth terms, prove that the sixth term will be a geometric mean between the fourth and ninth terms.

158. If a, b, c and d be all positive, prove that $\frac{(a+b)cd}{ad+bc}$ cannot be $>\frac{ac+bd}{a+b}$.

159. If the $n+1$ numbers a, b, c, d ... etc., be all different, and each of them a prime number, prove that the number of different factors of the expression $a^m bcd$ ... etc., will be $(m+1)2^n-1$.

160. If c_r denote the combinations of n things r at a time, prove that $c_1+2c_2+3c_3+\ldots+nc_n=n2^{n-1}$.

161. Prove that the co-efficient of x^n in the expansion of $\frac{1}{1+x+x^2}$ is 1, 0, or -1, according as n is of the form $3m$, $3m-1$, or $3m+1$.

162. There are 8 volumes of one work, 3 of another, and 1 of another. If they are placed on a shelf at random, find the chance that the volumes of the same work shall all be together.

163. Prove that $(1+x)^n=2^n\left\{1-n\frac{1-x}{1+x}+\frac{n(n+1)}{\underline{|2}}\left(\frac{1-x}{1+x}\right)^2-\ldots\right\}$.

164. Convert $\sqrt{10}$ into a recurring continued fraction, and find the first 5 convergents.

165. Find the value of $\frac{2}{x^2-1}-\frac{1}{x-1}$ when $x=1$.

166. Transform the equation $x^3-12x^2+3x+16=0$ into another wanting the second term.

167. Solve the equations:

(i.) $x^3-6x=6$,

and (ii.) $x^3+9x-6=0$, by Cardan's Method.

168. Get rid of the second term from the equation
$x^3-6x^2+18x-22=0$ by transformation.

169. Find all the roots of $x^3+3x^2-6x-8=0$.

170. Two detachments set out together to a station 30 miles away; but one detachment, by travelling a quarter of a mile an hour faster than the other, arrived there an hour sooner: find their rates of marching.

171. Find the least number which, when divisible by 39 and 56, shall leave the remainders 16 and 27 respectively.

172. Divide 100 into two parts, so that one may be divisible by 7 and the other by 11.

173. Solve $xyz=231$, $xyw=420$, $xzw=660$, and $yzw=1540$.

174. Sum the following series to ∞:—

$$\text{(i.) } \frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ \ldots$$

$$\text{(ii.) } \frac{3}{4}-\frac{9}{16}+\frac{27}{64}-\frac{81}{256}+\ \ldots$$

175. Find the number of shot in a complete pyramidal pile on a square base, each side of which contains 50 shot.

176. Find the number of shot in a complete pile on a rectangular base consisting of 25 tiers, the number of shot in the top row being 35.

177. Sum $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ \ldots$ to $2n-1$ terms and to ∞.

178. Find the radix of the scale in which the ordinary number 6629 will be expressed by 50405.

179. The sum of three numbers in H. P. is 13, and the sum of their squares is 61: find them.

180. Compare the co-efficient of the seventh term of the expansion of $(1+x)^{n+2}$ with the co-efficient of the seventh term of the expansion of $\frac{1}{(1-x)^{n-2}}$; and find n when the greater of these co-efficients is double the less.

181. Sum $\frac{1}{6.10}+\frac{1}{10.14}+\frac{1}{14.18}+\ \ldots$ to 10 terms.

182. Multiply $1+2x^{\frac{1}{4}}+3x+4x^{\frac{3}{4}}$ by $1-2x^{\frac{1}{4}}+3x-4x^{\frac{3}{4}}$.

183. Solve the equations :

$$\text{(i.)}\ \sqrt{x}+\sqrt{x-64}=\frac{96}{\sqrt{x-64}};$$

$$\text{(ii.)}\ x(y+z-x)=39-2x^2,\ y(x+z-y)=52-2y^2,\ z(x+y-z)=78-2z^2.$$

184. Determine whether the following pairs of numbers are prime one to another : (i.) 768, 823 ; (ii.) 627, 1034 ; (iii.) 30303, 12780.

185. Solve the equation $x^4=300x+301$.

186. If the roots of $x^4+ax^3+bx^2+cx+d=0$ be in H. P., prove that $b^2=4ac$.

187. Prove that

$$(1+2x+2x^2+\ .\ .\ .\ \text{to}\ \infty)^2=1+4x+8x^2+\ .\ .\ .\ +4nx^n+\ .\ .\ .\ \text{to}\ \infty.$$

188. Prove that

$$1-n^2+\frac{n^2(n^2-1)}{2^2}-\frac{n^2(n^2-1)(n^2-2^2)}{2^2 \cdot 3^2}\ .\ .\ .\ =0.$$

189. Sum the series

$$\text{(i.)}\ \frac{1\cdot 2}{3}+\frac{2\cdot 3}{3^2}+\frac{3\cdot 4}{3^3}+\ .\ .\ .$$

and (ii.) $\frac{1}{3}+1+2+3\frac{1}{3}+5+\ .\ .\ .$ each to n terms.

190. If a be the arithmetic mean between b and c, and b the geometric mean between a and c, prove that c will be the harmonic mean between a and b.

191. Divide $x^3-3a^2x-2a^3$ by $(x+a)^2$, and hence find a positive algebraical quantity which, substituted for x, shall make $x^3-3a^2x-2a^3=0$.

192. If the roots of the equation $x^2-ax+b=0$ be the cubes of the roots of the equation $x^2-cx+d=0$, find a, b in terms of c, d.

193. If the number of combinations of n things 2 together $=m$, prove that the number of combinations of m things 2 together $=$ thrice the number of combinations of $n+1$ things 4 together.

194. Write down the co-efficients of x^r in the expansions of $(1-x)^{-2}$ and $(1-x)^{-\frac{1}{2}}$.

195. Two vessels contain mixtures of wine and water ; in one there is thrice as much wine as water, in the other five times as much water as

wine ; find how much must be drawn off from each to fill a third vessel which holds 7 gallons, in order that its contents may be half wine and half water.

196. Find the n^{th} term and the sum of the first n terms of

$$\text{(i.)}\ \frac{1}{1.6}+\frac{1}{2.7}+\frac{1}{3.8}+\ \ldots$$

$$\text{and (ii.)}\ 8-12+18-27+\ \ldots$$

197. Sum $2+x-x^2-5x^3-13x^4-\ \ldots$ to ∞.

198. Prove that $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$ is $>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, when a, b, c are any three unequal positive quantities.

199. If $x+y+z=0$, prove that

$$x^5+y^5+z^5+5xyz(yz+zx+xy)=0.$$

200. Two numbers a, b being given, two others a_1, b_1 are formed by the relations

$$3a_1=2a+b,\ 3b_1=a+2b\ ;$$

two more a_2, b_2 are formed from a_1, b_1 in the same manner, and so on : find a_n, b_n in terms of a, b, and prove that, if n increase indefinitely, a_n and b_n are ultimately equal.

201. Find the square root of

$$(5^{\frac{3}{2}}+9^{\frac{3}{4}})\div(5^{\frac{1}{2}}+9^{\frac{1}{4}}).$$

202. Find the radix of the scale in which the denary number 124·96 is expressed by 444·44.

203. Determine whether $2x^3$ is greater or less than $x+1$.

204. Substitute $y+1$ for x in the equation

$$x^3-2x^2-5x+6=0,$$

and solve the resulting equation.

205. Without solving the equation

$$6x^2-96x+378=0,$$

find the sum of its roots, their difference, and the sum of their squares.

206. Show that the cube of any integer n, is the sum of n consecutive odd numbers. Verify this result in the case of 81.

207. A bag contains 3 white and 7 black balls, what is the chance of drawing 1 white and 2 black balls in 3 trials, drawing one ball at a time, the balls being replaced after each drawing? What is the chance if the balls are not replaced?

208. Find the quotient and the remainder when $3x^3+17x^2+10x-14$ is divided by $x-4$.

209. Increase by unity each root of the equation

$$x^3-5x^2+6x-3=0.$$

210. A rifleman fires point blank at a target 500 yards distant from him, and hears the bullet strike the target $4\frac{1}{3}$ seconds after he fires: an observer standing at a distance of 400 yards from the target, and 650 yards from the shooter, hears the shot strike the target $2\frac{1}{3}$ seconds after he hears the report of the rifle: find the velocity of the bullet and that of sound, supposing both velocities uniform.

211. Express the co-efficient of the tenth term of $(1+x)^{\frac{1}{2}}$ as a fraction in its lowest term, with a power of 2 for its denominator.

212. Given $x+y=\sqrt{m}$ and $x-y=\sqrt{n}$, express x^3+y^3 in terms of m and n; and show that

$$16(x^4-7x^2y^2+y^4)=(5m-n)(5n-m).$$

213. Solve the equations $x(y+z)=14$, $y(z+x)=18$, $z(x+y)=20$.

214. Find the number of shot in a complete pyramidal pile (i.) on a square base, (ii.) on a triangular, each side of the base in both cases containing 13 balls.

215. Find the quotient and the remainder when

$3x^5-7x^3+8x^2-10x+53$ is divided by $x+3$.

216. In how many ways can a party of 3 ladies and 3 gentlemen be formed from 7 ladies and 11 gentlemen, so that a certain couple may never be together in the party?

217. Find the $(r+1)^{th}$ term of $(1-x)^{-4}$ in its simplest form.

218. Prove that the following series are *convergent*:—

$$\text{(i.) } x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ \ldots \text{ if } x \text{ be } <1;$$

$$\text{and (ii.) } 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ \ldots \text{ whatever } x \text{ may be.}$$

219. Solve the equations:

$$\text{(i.)}\ \sqrt[3]{a+x}+\sqrt[3]{b-x}=\sqrt[3]{a+b}:$$

$$\text{(ii.)}\ x^2+\left(\frac{a^2}{b}+\frac{b^2}{a}\right)x+ab=0.$$

220. If $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$ and $\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=0$, prove that

$$(b+c)x+(c+a)y+(a+b)z=0.$$

221. Find the factors of $alx^2+(am+bl)xy+bmy^2+clxz+cmyz$.

222. If $x+h$ be substituted for x in the general equation $F(x)=0$, find the value of h that the second term may vanish.

223. Find the first term of the fourth order of differences of the series 1, 8, 27, 64, 125.

224. Find the first terms of the first three orders of differences of the series $a, b, c, d, e \ldots$; and show that the first term of the n^{th} order

$$=(-1)^n\left\{a-nb+\frac{n(n-1)}{2}c-\text{ to } n \text{ terms}\right\}.$$

225. If $y=x-x^2+x^3-x^4+\ .\ .\ .$ *reverse* the series; that is, prove that

$$x=y+y^2+y^3+y^4+\ .\ .\ .$$

ANSWERS.

EXAMPLES. I.

2. $a^3 + \ . \ . \ . \ + 3a^2b + 3a^2c + \ . \ . \ . \ + 6abc : a^3 - b^3 - c^3 - 3a^2b - 3a^2c - 3b^2c + 3b^2a + 3c^2a - 3c^2b + 6abc$.

4. $3a + b + 2c + d$.

6. $x^8 - 2x^6 + x^4 + 4x^3 - 1$.

7. $(x^n - y^m)(x^{2n} + x^ny^m + y^{2m})$.

8. $x^4 - 4x^2yz + 7y^2z^2$.

10. $x^4 + 2x^2 + 6 + \dfrac{2}{x^2} + \dfrac{1}{x^4}$.

11. $(x-a)\left(x+\dfrac{1}{a}\right) : (x-4)(x-1)(x+1)(x-2)$.

13. $8 + 12a + 18a^2 + 27a^3$.

14. 0. [See § 8, p. 2.]

15. $(x-y+1)(x^2+y^2+1+y-x+xy)$.

17. $1 - 4x - \dfrac{46}{15}x^2 + \dfrac{10}{3}x^3$.

18. $x^3 + (bc + \ . \ . \ . \ - a^2 - \ . \ . \ . \ )x + (a-b)(b-c)(c-a) : x^3 - 3(a-b)^2x - 2(a-b)^3$.

19. $2(b+c)(c+a)(a+b)$.

20. $4(x+y+z)$.

21. $16x^2(1-4x^2)$.

22. $x^2 + (a-b)x - ab$.

23. $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.

24. $(a+b)(c+d)$.

27. $(a-b)(b-c)(c-a)$.

28. $(m+1)b^2x^2+(m+1)(n+1)abx+(n+1)a^2$.

29. $x^2y^2(a^4+b^4)-a^2b^2(x^4+y^4) : 2(a^2+b^2)(x^2+y^2)$.

30. $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$.

31. $x^3-3x^2y-3xy^2+y^3$.

32. $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$.

33. $x-d$.

34. $m^2-n^2-p^2-q^2-2np-2nq-2pq$.

35. $(x-m+n)(x-m-n) : (x-y)(y-z)(z-x)$.

36. $x^3+2x^2y-23xy^2-60y^3 : x^4-15x^2+10x+24$.

37. $x^3-\frac{7}{36}x+\frac{1}{36}$.

38. $\left(x^4+\frac{1}{16}\right)\left(x^2+\frac{1}{4}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$.

39. $2ax-(3b-4c)y$.

40. $bc(b-c)+ca(c-a)+ab(a-b)$.

41. 0.

44. $(x+y)^2+4(x+y)+16$.

46. 42,000.

47. $9x-14$.

48. $a^{12}+3a^{10}b^2+2a^9b^3+3a^8b^4+6a^7b^5+2a^6b^6+\ldots+b^{12}$.

50. x^5-2x^4.

51. $x^4-2x^3-13x^2+14x+24$.

52. $4a^2+9b^2+\frac{c^2}{4}+\frac{d^2}{16}+e^2+12ab-2ac-ad+4ae-3bc-\frac{3bd}{2}+6eb+\frac{cd}{4}-ce-\frac{de}{2}$;

$a^3+b^3+\ldots+3a^2(b+c+d)+3b^2(c+d+a)+\ldots+6(bcd+cda+\ldots)$.

53. $a^2+b^2+c^2$.

54. $2a-b+3c$.

55. $x^7-x^6y+x^5y^2-\ldots-y^7 : x^7+x^6y+x^5y^2+\ldots+y^7$:
$x^4-x^3y+x^2y^2-xy^3+y^4$.

56. $ab-acd+acef-acegh+acegk$.

57. 8200.

59. $9xy^2$.

60. $(2x-y)(m^2-4n^2):9$.

EXAMPLES. II.

1. $x+3$.
2. x^2-a^2.
3. x^2-1.
4. $x^2-(p-q)x+q^2$.
5. $(a-1)x+a$.
6. $x-3$.
7. $5x^2-1$.
8. $x-15$.
9. $(x-a)^2$. [See p. 8, § 6.]
10. $(x-1)^2$.
11. $2x^2+5xy+3y^2$.
12. $3x^2-4xy-y^2$.
13. $x^6-2x^5+4x^4-16x^2+32x-64$.
14. a^4-b^4.
15. $(x+2y+3z)(2y+3z-x)(3z+x-2y)(x+2y-3z)$.
16. x^6-y^6.
17. $(2x-1)(3x+1)(4x-3)$.
18. $3(x^2-4a^2)(x-3a)^2$.
19. $105x^2y^2(x^2-y^2)^2$.
20. x^2-3x+1.
21. x^2+xy+y^2.
22. x^2+2x+3.
23. $\dfrac{3x-a-2b}{x^2-(a+b)x+ab}$.

24. $\dfrac{3a-2x}{5a+3x} : \dfrac{x+3}{x-3}$.

25. $\dfrac{x(a-b)-y(a+b)}{x(a-b)+y(a+b)}$.

26. $\dfrac{x^2y^2+xyz^2+z^4}{xy-z^2}$; $\dfrac{x^2y^2-xyz+z^2}{x^4y^4-x^3y^3z+x^2y^2z^2-xyz^3+z^4}$.

27. $\dfrac{a+d}{f+2x}$.

28. $\dfrac{3x^2-ax+a^2}{2x^2+a^2}$.

30. $\dfrac{3x^2+7x-12}{(x^2-9)(x^2-16)} : \dfrac{1}{(x-3)(x-4)}$.

31. $\dfrac{ax(x^2-ax+a^2)}{(a-x)^2(x^2+ax+a^2)}$.

34. 1.

36. $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2b^2}$.

37. $\dfrac{2x^2-x^3}{x-1}$

38. (i.) $x^a+y^b+z^c$; (ii.) $x-a$.

40. $a-b$.

41. $(2x-3a)$; $(2x-3a)(3x+2a)(2x^2+2ax+3a^2)$.

42. $(\alpha)\dfrac{x+3y}{x-3y}$; (β) $\dfrac{x^2-b^2}{x^3-a^3}$.

43. $x-8a : 3x+5$.

44. $-\dfrac{2}{x-3}$.

45. (i.)$\dfrac{x-3}{x+3}$; (ii.) $4x^2+y^2+9z^2+3yz-6zx+2xy$.

46. (i.) $\dfrac{2a-b}{a^2-1}$; (ii.) $\dfrac{1}{x^2-1}$; (iii.) $\dfrac{ab(1-x)}{x(b^2-a^2)}$.

47. x^6-y^6.

48. (i.) $a^2+xy : (a^2+xy)(2x+3y)(2x-3y)$; (ii.) $x-1 : (x-1)(x-2)(x+2)(x-3)(x+3)$.

49. (i.) $\dfrac{3n^2}{(3m+2n)(9m^2-n^2)}$; (ii.) $\dfrac{x+6}{x-5}$.

50. $\dfrac{x(x+1)}{x^2+4x+1}$.

51. (i.) $\dfrac{3}{x^3-1}$; (ii.) $\dfrac{y}{x+y}$.

52. $2a-b-1=0$, $3a-b-c=0$; $a=-74$, $b=120$.

53. (i.) $\dfrac{x^6+y}{x^2-y^2}$; (ii.) $\dfrac{x+b}{bx+1}$.

55. 2.

56. $a=0$; $b=-36$.

57. $x-3$.

58. $x-2$.

59. $45b-b^2-7bc-36c+12c^2-c^3=0$.

60. $x^6-2x^5+x^4-x^2+2x-1$.

EXAMPLES. III.

1. $x^2-x+\dfrac{1}{4}$.

2. $1-2a+3a^2$.

3. $\dfrac{1+3x}{1-x}$.

4. $x+4-\dfrac{8}{x}$.

5. 724.

6. (i.) $\dfrac{x}{y^2}-\dfrac{y^2}{x}$; (ii.) $\dfrac{a}{2}+\dfrac{2}{3a^2}$; (iii.) $x-4+\dfrac{2}{x}$.

7. $2x-3y$.

8. ·0785 ; 5520 ; 8·78 ; 79·5 ; 1·02003.

9. 99·8 ; ·696 ; 376 ; 428 ; 309 ; 807 ; 878 ; 795.

10. $x+\dfrac{1}{2}-\dfrac{1}{4x}$.

11. $a+b$.

12. $12\frac{1}{24}$; $12\frac{12}{13}$; $15\frac{1}{30}$; $5\frac{23}{44}$.

13. (i.) $x^2+2ax-a^2$; (ii.) $x^3-11x+17$; (iii.) $x+3y+2z$.

14. $x-\dfrac{1}{x}+1$.

15. 10205; ·17382.

16. $\dfrac{x^2}{4}+\dfrac{3x}{8}+\dfrac{1}{2}$.

17. $2y$.

18. $(x-1)(x-2)(x-3)$.

19. a^6-b^6; $x+3y+2z$.

20. $5-\dfrac{3y}{4x}-\dfrac{2x}{7y}$.

EXAMPLES. IV. (A.)

1. 6, -2.

3. $P=p^4-4p^2q+2q^2$; $Q=q^4$.

5. (i.) $x^2+10x+61=0$; (ii.) $x^2+x+1=0$.

7. $m=\dfrac{bc+ca+ab-a^2-b^2-c^2}{3}$.

9. 18 *only*, and not 3.

10. $b^2=ac$.

EXAMPLES. IV. (B.)

1. 3.

2. $\dfrac{b^2-4a^2}{4a}$, 0.

3. $\dfrac{2ab}{b^2+1}$.

4. ± 5.

5. $\pm 2, \pm 3$.

6. $2, -1$.

7. $2, \dfrac{1}{2}, \dfrac{1 \pm \sqrt{-24}}{5}$.

8. $6\dfrac{2}{7}$.

9. $\dfrac{1}{\sqrt[3]{4}}, 1$.

10. $2(\sqrt{3}-2), 2-\sqrt{3}$.

11. $-a, -b$.

12. $4, -5$.

13. $0, 2, 7$.

14. $1, 2, 3, \dfrac{-11 \pm \sqrt{-23}}{12}$.

15. $x=6$; $y=2$.

16. $x=7, 5$; $y=5, 7$.

17. $x=25, 49, 107 \pm 12\sqrt{71}$;
$y=49, 25, 107 \mp 12\sqrt{71}$.

18. $x=\pm 2, \pm \dfrac{1}{\sqrt{3}}$; $y=\pm 1, \mp \dfrac{5}{\sqrt{3}}$.

19. $x=-3 \pm \sqrt{3}, 3, 2$; $y=-3 \mp \sqrt{3}, 2, 3$.

20. $x=3, -\dfrac{5}{4}$; $y=\dfrac{3}{4}, 5$.

21. $x=\dfrac{2abc}{ca+ab-bc}$; $y=\dfrac{2abc}{ab+bc-ca}$; $z=\dfrac{2abc}{bc+ca-ab}$.

22. $x=\pm\sqrt{2}$; $y=\pm\sqrt{3}$; $z=\pm\sqrt{6}$.

23. $x=\pm\dfrac{31}{120}$; $y=\mp\dfrac{319}{120}$; $z=\mp\dfrac{481}{120}$.

24. $x=2, 3, \dfrac{5 \pm \sqrt{-359}}{2}$; $y=3, 2, \dfrac{5 \mp \sqrt{-359}}{2}$; $z=4, -14$.

25. $4, 3$.

26. $x=9$; $y=4$.

27. $x=a$; $y=b$.

28. 2, -3.

29. 3.

30. $x=\pm\frac{9}{2}\sqrt{\frac{5}{11}}$; $y=\pm 7\sqrt{\frac{5}{11}}$; $z=\pm\frac{21}{2}\sqrt{\frac{5}{11}}$.

31. $2\frac{1}{7}$.

32. 3, $-\frac{7}{22}$.

33. $x=8, 3$; $y=2, 7$.

34. 2·25.

35. 9, -3.

36. $x=b, \frac{b}{\sqrt{5}}$ $\quad y=a, -\frac{a}{\sqrt[3]{5}}$.

37. $\frac{2\frac{1}{2}}{5}, \frac{3}{10}$.

38. $x=-2, +1$; $y=\frac{1}{2}, -1$.

39. 7.

40. $-\frac{b}{a-b}, \frac{a}{a+b}$.

41. $x=5$; $y=7$.

42. 4, $-4\frac{1}{6}$.

43. $\frac{7}{2}$, 10, -3.

44. $\frac{ca}{b}$.

45. $b, -\frac{ab}{a+b}$.

46. $x=\pm 3$, $y=1$.

47. $\frac{a-7b\pm(a+b)\sqrt{33}}{8}$.

48. $5, \ -2$.

49. $x = \pm \dfrac{3+2\sqrt{2}}{\sqrt{3(\sqrt{2}+1)}}$; $y = \pm \dfrac{1}{\sqrt{3(\sqrt{2}+1)}}$.

50. $x = \pm 3$; $y = \pm 4$; $z = \pm 6$.

51. $x = 1, 3$; $y = 2, \ -6$.

52. $\dfrac{4}{3}$.

53. 2.

54. $3, \ -\dfrac{3}{5}$.

55. $x = \pm 2, \pm \dfrac{6}{\sqrt{5}}$; $y = \pm 2, \pm \dfrac{2}{\sqrt{5}}$.

56. $\dfrac{1}{2}$.

57. 0.

58. 10.

59. $x = 5$; $y = 4$.

60. $x = \pm 5, \ \pm 4$; $y = \pm 4, \ \pm 5$.

61. 4.

62. $x = \pm 8, \ \pm 24$; $y = 6, \ 18$.

63. $x = 3, \ 2, \ -3 \pm \sqrt{3}$; $y = 2, \ 3, \ -3 \mp \sqrt{3}$.

64. $1, \dfrac{3 \pm \sqrt{5}}{2}$.

65. $\left(\dfrac{2ac}{bc^2+b}\right)^{\frac{1}{n}}$.

66. $\pm \sqrt{2ab-b^2}$.

67. $\dfrac{\sqrt{a}}{\sqrt{a+2}}$.

68. $\dfrac{(a-b)^2}{2b}$.

69. 2.

70. $-11\frac{15}{16}$.

71. $\dfrac{9a}{16}$.

72. $\dfrac{b^2-a^2}{4a-b}$.

73. $\dfrac{p}{4}, \dfrac{3p}{4}$.

74. $x=8\dfrac{4}{5}$; $y=-11$.

75. $x=-3, 2, -2\pm\sqrt{2}$; $y=+3, 3, -1\pm\sqrt{2}$.

76. $x=a+b$; $y=a-b$.

77. $10, -1, \dfrac{9}{2}$.

78. $\dfrac{53}{67}$.

79. $\dfrac{cd(a+b)-ab(c+d)}{ab-cd}$.

80. $1, \dfrac{1}{9}$.

81. $x=\pm 10$; $y=\pm 4$.

82. $x=\pm 4$; $y=\pm 3$.

83. $x=12, 3$; $y=3, 12$; $z=6$.

84. $x=2, 3$; $y=3, 2$.

85. $64, -\left(\dfrac{97}{3}\right)^{\frac{3}{2}}$

86. $243, (-28)^{\frac{5}{3}}$.

87. $\pm 64, \pm\left(\dfrac{37}{3}\right)^{\frac{3}{2}}$.

88. $8, \left(-\dfrac{5}{2}\right)^{\frac{3}{2}}$.

89. $\pm 4, \pm\sqrt{\dfrac{-31}{2}}$.

90. $x=3$; $y=2$; $z=1$.

91. $x=3$; $y=6$.

92. 6, 10.

93. 3, 0.

94. $x = \pm 5, \pm\frac{13}{2}\sqrt{\frac{11}{70}}$; $y = \pm 3, \pm\frac{31}{2}\sqrt{\frac{11}{70}}$.

95. $x=6, 3, 2\pm\sqrt{-14}$; $y=3, 6, 2\mp\sqrt{-14}$.

96. $x=a$; $y=-b$.

97. $x=\pm 9$; $y=\pm 3$.

98. $x=\pm 3$; $y=\pm 2$; $z=\pm 1$.

99. $x=\pm 3, \pm\frac{3\sqrt{2}}{2}$; $y=\pm 1, \pm 4\sqrt{2}$.

100. $2, -3\frac{1}{5}$.

EXAMPLES. IV. (C.)

1. 24.
2. 38.
3. 14 yards and 42 yards.
4. 13.
5. 1s. 4d.; 1s. 2d.; 10d.
6. $3\frac{1}{3}$ per cent.
7. £187, 10s. 0d.
8. 1s. 4d.; 2s. 10d.
9. 11s.; 9s.
10. 300.
11. 6, 12.
12. 41, 40, 9.
13. 20 sovereigns.
14. 40 miles per hour.
15. $\frac{5}{3}, \frac{36}{5}$.
16. 40 and 45 miles an hour.

17. 14 lbs. for 3d., and 2d. for every additional 7 lbs.
18. £520 ; £260.
19. 1676.
20. 25 per mile.
21. 10 gallons.
22. 2, 5, 8.
23. 4 minutes.
24. $32\frac{8}{11}$ minutes past 3 o'clock.
25. 3 miles an hour.
26. 40 hours, 60 hours.
27. 75 shares at £25 each.
28. £600, £900.
29. £130, £260.
30. 36, 12.
31. 128 inches, $1\frac{1}{2}$ inches.
32. 4, 16, 12, 16.
33. 4, 8.
34. Man £10 ; woman £8 ; boy £6 ; girl £2.
35. 72.
36. £9 for an ox ; £1, 10s. for a sheep.
37. $4\frac{43}{52}$ yards.
38. 2 : 1.
39. 4·68559.
40. $27\frac{3}{11}$ minutes past 8 o'clock.
41. 1 shilling a dozen.
42. 7, 21, 2, 98.
43. 54, 81.
44. 150 miles.
45. 24 lbs.

46. $5\frac{5}{11}$ minutes past 1 o'clock.

47. 12 feet, 15 feet.

EXAMPLES. V.

4. 11 : 24.

7. 20 miles an hour.

11. 10 cwt.

18. 29 : 25.

19. $x = \frac{112}{z}$.

20. 10, 2.

21. 121 : 169 ; 7 : 20 ; 2 : 5.

22. $\frac{2}{3} > \frac{5}{8} > \frac{11}{21} > \frac{72}{168}$.

EXAMPLES. VI.

1. $a^{5m-2n} - a^{2m+n}$.

2. $\frac{x^{n-1}y}{(x+y)^m}$.

3. $\frac{ba^n + ca^p + da^q - e}{a^m}$.

4. $x^{m-n}y^{m+n}$.

5. $\frac{4a^m b^m}{a^{2m} - b^{2m}}$.

6. ab.

8. $x^3 - 4y^3 - 9z^3 - 12y^{\frac{3}{2}}z^{\frac{3}{2}}$.

9. $\frac{x^{3n}b + a^n b^3}{a^{6n} - 2a^{5n}b + 3a^{4n}b^2 - 3a^{3n}b^3 + 3a^{2n}b^4 - 2a^n b^5 + b^6}$

10. $a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - c^{\frac{1}{3}}a^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}}$.

11. (α.) $\frac{625a^5}{384b^2}$; (β.) $abcde$; (γ.) $\frac{27c^5d^5}{200a^5b^5}$.

12. $(5a-7b)^n$.

13. $x^{m+2}-x^{m+3}y^{-1}-x^{m+6}y^{-4}+x^{m+7}y^{-5}$.

14. 1.

15. $5\sqrt{a}-3\sqrt{b}$; $4+\sqrt{2}$.

16. $\dfrac{1+2y^{-\frac{1}{2}}}{1-y^{\frac{1}{2}}}$.

17. $\dfrac{1}{2}x^{\frac{1}{2}}a^{-\frac{1}{2}}-a^{\frac{1}{2}}x^{-\frac{1}{2}}-\dfrac{1}{3}$.

18. $3\sqrt{m}-5\sqrt{n}$; $p+\sqrt{p^2+q^2}$; $\sqrt{p+q}+\sqrt{p-q}$.

20. 18.

22. $(1+x^2)(1+\sqrt{3}x+x^2)$.

23. 91.

24. $3\sqrt{5}-2\sqrt{3}$; $5+\sqrt{6}$.

27. $\dfrac{\sqrt{2}+2\sqrt{3}+3\sqrt{6}-4}{4}$.

28. $\dfrac{1}{1-x^2}$.

29. $5-2\sqrt{6}$.

30. $\dfrac{\sqrt{3}}{2}-\dfrac{\sqrt{3}}{2\sqrt{2}}$: $\sqrt{51}-7$.

31. 0.

32. $(x^2-ax+a^2)(x+\sqrt{-3ax}-a)$.

33. 10.

34. $\dfrac{(6a^2b-2b^3)\sqrt{-1}}{a^2+b^2}$.

35. 0.

36. -12.

37. $\dfrac{3x^2}{x^3-1}$.

39. 1, $\sqrt{-1}$, -1, $-\sqrt{-1}$.

40. 0.

41. (i.) $\dfrac{6}{3^{\frac{2}{3}}-1}$; (ii.) $-3-2\sqrt{2}$.

42. $2x-\dfrac{1}{2x}$.

43. $\frac{1}{2}(\sqrt{2}+\sqrt{3})$.

44. $3+\sqrt{7}$.

45. ·447.

46. $\dfrac{16+5\sqrt{7}-\sqrt{5}\sqrt[4]{7}(5+\sqrt{7})}{9}$.

47. Rationalising factor is $3^{\frac{5}{2}}-3^{2}2^{\frac{1}{3}}+3^{\frac{3}{2}}2^{\frac{2}{3}}-3.2+3^{\frac{1}{2}}2^{\frac{4}{3}}-2^{\frac{5}{3}}$; result $=19$.

48. $2(63+15\sqrt{2}-21\sqrt{5}-5\sqrt{10})$.

49. 1.

50. $2(bc+ca+ab)-a^2-b^2-c^2$.

51. $-3+4\sqrt{-1}$.

52. $14+2\sqrt{21}$; $6+\sqrt{7}$.

53. $\dfrac{6\sqrt{10}}{5}$.

54. $\sqrt{1+a}+\sqrt{1-a}$.

55. $\sqrt{2x-3}+\sqrt{x+2}$.

56. $1+\sqrt{-1}$.

57. $2a$.

59. $\sqrt{4c^2-d^2}\pm\sqrt{ab}$.

60. 6 ; 13 ; 8.

63. $x^2-\sqrt{2}\,.\,x+\sqrt{3}$.

64. $2+\sqrt{7}$.

65. $x+1+\dfrac{1}{x}$; $x-x^{\frac{1}{2}}$.

67. $\sqrt{1+x^2}+\sqrt{1-x^2+x^4}$.

68. $\dfrac{35+12\sqrt{6}}{19}$.

69. $\frac{1}{7}$.

70. $1 \cdot 166$.

71. (i.) $\sqrt{2}+\sqrt{3}+\sqrt{7}$; (ii.) $3+\sqrt{2}+\sqrt{11}$.

72. $3+\sqrt{5}$.

75. $\dfrac{2}{x}$.

EXAMPLES. VII.

1. 0; $\dfrac{2119}{1558}$.

2. n^3.

3. $1 \cdot 1n^2 + \cdot 2n$.

4. 14.

5. 109,225; $\dfrac{85}{128}$.

6. $\dfrac{5}{7}$; $\dfrac{(a+x)^3}{4ax(a-x)}$.

7. $1, \dfrac{6}{5}, \dfrac{3}{2}$ $(2, 3, 6)$, ∞, $-6, -3$.

8. $\dfrac{6}{5}, \dfrac{3}{4}, \dfrac{6}{11}, \dfrac{3}{7}, \dfrac{6}{17}, \dfrac{3}{10}$.

9. 24.

10. (i.) $\dfrac{n(3n+1)}{2}$; (ii.) $\dfrac{n(9-7n)}{2}$; (iii.) $15\frac{2047}{2048}$; (iv.) $\frac{3}{10}$; (v.) $-31\frac{1}{6}$; (vi.) $44\frac{1845}{2048}$.

12. $n(a+b-2nb)$; the greatest integer in $\dfrac{2a}{b}$; a.

14. $\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{1}{12}$, etc.

19. 13 years.

20. G. M. $=1-x^4$; $a=(1+x)^2(1+x^2)$, $b=(1-x)^2(1+x^2)$.

21. 2, 3, 4, 5.

22. 3, 5, 7, 9, 11, 13.

23. 1, 2, 3, 4, 5, 6, 7.

24. 6, 8, 10, 12.

26. 2, 3, 6.

30. 4 or 13.

33. 1, $7 \pm 4\sqrt{3}$.

34, $d = 10$; 4989.

36. (i.) $290\frac{2}{3}$; (ii.) $\dfrac{n-1}{2}$.

37. 0 ; 1728.

39. (i.) 75 ; (ii.) $1\frac{1}{5}$.

40. 4 : 1.

41. 2550 yards.

42. 3, 5, 7.

44. $(2n+1)(a+2nd)$.

46. 1, 4, 7, 10.

48. 8, 12, 16, 20.

49. 4.

50. $1\frac{5}{7}$.

51. 1.

52. 1 and 49.

54. $\frac{2}{9}$.

55. 2, 4, 6.

56. 4 or 7.

59. 3, 5, 7 . . . 17.

EXAMPLES. VIII.

1. 297,000.

2. 1120 ; 60.

4. 1260.

5. 56.

6. 9,979,200.

7. $\dfrac{m(m-1)(m-2)}{6}$; 20.

8. 15, 6.

9. 35.

10. 10.

11. 210.

12. 658.

13. 8.

14. 210.

15. 19,958,400.

16. 80.

17. 3.

18. (i.) 7140 ; (ii.) 42840.

19. 14112 ; 4410.

20. $\dfrac{\underline{|21}}{2\,\underline{|3}\,\underline{|4}\,\underline{|5}\,\underline{|10}}$.

21. 7.

22. 55.

24. 1680.

25. 3.

26. 35 to 1.

27. $\dfrac{2ab}{(a+b)^2}$.

28. $\frac{1}{343}$; $\frac{216}{343}$.

29. $\frac{1}{8}$.

30. 6s. $10\frac{1}{2}$d.

31. $\frac{7}{15}$.

32. $\frac{16}{5525}$.

33. $\frac{16}{25}$.

34. $\frac{175}{429}$.

35. $\frac{2619}{3125}$.

36. $\frac{4}{9}$.

37. $\frac{1}{270725}$; $\frac{16}{5525}$.

38. $\dfrac{|\underline{13}\;|\underline{39}}{|\underline{52}}$.

39. $\frac{1}{6}$; $\frac{5}{36}$; $\frac{1}{9}$.

40. $\frac{20412}{78125}$.

41. $\frac{1}{18}$.

42. $\frac{1}{26}$.

43. $\frac{23}{648}$.

44. $\frac{11}{36}$.

45. Each once *only*, 149 to 13 ; otherwise 293 to 31.

46. 11 : 4.

47. $\frac{2}{3}$; $\frac{29}{30}$.

48. *ab* ; *ab*.

49. $\frac{3781}{46656}$.

50. £1, 12s. $1\frac{1}{5}$d.

51. $\frac{496}{729}$.

52. $\frac{2560}{28561}$.

53. $\frac{1}{525}$; $\frac{1}{151200}$.

EXAMPLES. IX.

1. 62·48.
2. 101211.
3. 252710.
4. 12312331.
5. *e7t8*.
6. 9294 ; 344.
7. 122·2.
8. 1110111001111 ; *n1e*.
9. *eee*.

10. $3^7 - 3^6 - 3^5 + 3^4 + 3^3 - 3 - 1.$

11. $1\frac{17}{49}.$

12. 15.

13. 7.

14. $\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$ to $\infty = 1.$

15. 43 in scale of $7 = 34$ in scale of 9.

16. 4112.

17. $3^5 + 3^4 + 3^2 - 3 - 1$

18. 5.

19. 248.

21. 525.

EXAMPLES. X.

1. $330x^7.$

2. $1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3.$

3. $-\frac{14}{3^{11}}x^{10}$; $-48384a^5x^3.$

4. 15 ; $\frac{243}{2^{10}}.$

6. $\frac{7 \,.\, 10 \,.\, 13 \ldots (3r+1)}{\underline{|r-1}}\left(\frac{2x}{3}\right)^{r-1}.$

7. $1 + \frac{3}{2}x^2 + \frac{15}{8}x^4 + \frac{35}{16}x^6 + \frac{315}{128}x^8.$

8. (i.) 20 ; (ii.) 126.

10. $1 - \frac{1}{2}x + \frac{1 \,.\, 3}{2^2\underline{|2}}x^2 - \ldots$; $(-1)^r.\frac{1 \,.\, 3 \ldots (2r-1)}{2^r\underline{|r}}.$

11. $a^2 + 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab + 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2$; $-105x^2y^{13} + 15xy^{14} - y^{15}.$

12. $1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6$; $-\frac{28}{3^{r-1}}\times\frac{2.5.8\ldots(3r-10)}{|\underline{r-1}}x^{r-1}$.

13. (i.) If $\dfrac{n+1}{\frac{a}{x}+1}$ is an integer (p), then the p^{th} and $(p+1)^{th}$ terms are equal, and greater than any other; (ii.) if $\dfrac{n+1}{\frac{a}{x}+1}$ is not integral, let p be the greatest integer which it contains, then the $(p+1)^{th}$ term is the greatest. [See § 6, p. 61.]

15. $a^n+n\sqrt{-1}a^{n-1}b-\frac{n(n-1)}{|\underline{2}}a^{n-2}b^2-\frac{n(n-1)(n-2)}{|\underline{3}}\sqrt{-1}a^{n-3}b^3$.

16. $\dfrac{|\underline{2m}}{(|\underline{m})^2}$.

17. $-\frac{4}{125}a^{-\frac{11}{3}}b^{\frac{1}{2}}$.

18. $r+1$; $\dfrac{1.3.5\ldots(2r-1)}{2^r|\underline{r}}$.

20. $\dfrac{1}{na^{n-1}}$.

21. $1-2x-x^2-\frac{4}{3}x^3-\frac{7}{3}x^4$; $-\frac{2\times4.7.10\ldots(3r-5)}{|\underline{r}}x^r$.

23. See p. 82, § 4 (a).

24. $\dfrac{|\underline{16}}{(|\underline{8})^2}a^8x^8$.

26. $n=\frac{1}{3}$ or $-\frac{4}{3}$: co-efficients of fifth term $=\frac{35}{243}$ or $\frac{5}{243}$.

31. $\sqrt{\frac{2}{3}}$.

34. 1·0000250.

36. 462.

37. $x^m + nx^{m-1} + \dfrac{n(n+1)}{\underline{|2}}x^{m-2} + \text{ etc. };\ \dfrac{|n+m-1}{|m\,|n-1}$.

38. $4n+1$.

39. $3{\cdot}03687$; $4{\cdot}99936$.

EXAMPLES. XI.

1. $4+\dfrac{1}{2}\,\dfrac{1}{+1+1+2}$; $4, \frac{9}{2}, \frac{13}{3}, \frac{22}{5}, \frac{57}{13}$; $\dfrac{1}{2}\,\dfrac{1}{+7+1+6}$; $\frac{1}{2}, \frac{7}{15}, \frac{8}{17}, \frac{55}{117}$:

$$\frac{1}{13}\,\frac{1}{+1+3+7+11};\ \tfrac{1}{13}, \tfrac{1}{14}, \tfrac{6}{83}, \tfrac{43}{595}, \tfrac{479}{6628}:$$

$$8+\frac{1}{2}\,\frac{1}{+4+7+6};\ 8, \tfrac{17}{2}, \tfrac{76}{9}, \tfrac{549}{65}, \tfrac{3370}{399}.$$

2. $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$: $2, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}, \frac{2889}{1296}$: $6, 7, \frac{41}{6}$: $\frac{48}{7}, \frac{617}{90}, \frac{665}{97}$:
$3, 4, \frac{11}{3}, \frac{15}{4}, \frac{101}{27}, \frac{116}{31}$.

3. $$\cfrac{1}{a+\cfrac{1}{a+1+\cfrac{1}{a+2+\cfrac{1}{a+3}}}};\ \frac{1}{a}, \frac{a+1}{a^2+a+1}, \frac{a^2+3a+3}{a^3+3a^2+4a+3}.$$

4. $x=b^2-ab$.

6. 1.

7. $\sqrt{\dfrac{5}{3}}$.

EXAMPLES. XII.

1. $1+3x+4x^2+7x^3+\ldots$; $1-4x+5x^2-x^3\ldots$

2. $\dfrac{1}{(a-b)(a-c)}$, etc.

3. $\dfrac{2}{x+4}-\dfrac{2}{x+2}$; $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}$; $\dfrac{3}{x+1}+\dfrac{4}{(x+1)^2}-\dfrac{3}{x}$.

4. $\frac{2}{x-3}-\frac{2}{x-1}=2(1+x+x^2+\dots)-\frac{2}{3}\left(1+\frac{x}{3}+\frac{x^2}{3^2}+\dots\right)$;

$$\text{co-efficient of } x^r=2-\frac{2}{3^{r+1}}.$$

$$\frac{1}{x-1}-\frac{x}{x^2+3}=-(1+x+x^2+\dots)-\frac{x}{3}\left(1-\frac{x^2}{3}+\frac{x^4}{3^2}-\dots\right);$$

co-efficient of $x^r=-1$, or $-1+\frac{(-1)^{\frac{r+1}{2}}}{3^{\frac{r+1}{2}}}$, according as r is *even* or *odd*.

5. $\frac{1}{3^3}-\frac{1}{4^3}$; $\frac{1}{3^{r+1}}-\frac{1}{4^{r+1}}$.

6. $(1+x+x^2+\dots)+\left(1+\frac{x}{2}+\frac{x^2}{2^2}+\dots\right)-\left(1+\frac{x}{3}+\frac{x^2}{3^2}+\dots\right)$;

$$\text{co-efficient of } x^n=\left(1+\frac{1}{2^n}-\frac{1}{3^n}\right).$$

7. $\frac{7}{3(x-2)}-\frac{7}{4(x-3)}-\frac{13}{12(x+1)}+\frac{1}{2(x-1)}$;

$$\frac{1}{x}+\frac{2}{1-x}+\frac{1}{1-3x}.$$

8. $1+x+5x^2+13x^3\dots$

9. $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$; $\frac{1}{2(x+1)^2}+\frac{4}{x+1}-\frac{8x-7}{2(x^2+1)}$.

10. 1; $-\frac{b+c}{a}$; $\frac{(b+c)c}{a^2}$; $-\frac{(b+c)c^2}{a^3}$.

11. $1+(a+b)x+(a^2+ab+b^2)x^2+(a^3+a^2b+ab^2+b^3)x^3+\dots$;

$$\frac{1}{a-b}\left\{\frac{a(1-a^nx^n)}{1-ax}-\frac{b(1-b^nx^n)}{1-bx}\right\}.$$

EXAMPLES. XIII.

1. 2300.

2. $n(3n^2+6n+1)$; 11490.

4. (i.) $\frac{3^{n+1}-3-2n}{4\times 3^{n-1}}$; (ii.) $8\frac{2}{3}$.

5. $\dfrac{3^{n+2}-8n-9-2n^2}{4\times 3^n}$.

6. 1540 ; $\dfrac{2n(n+1)(2n+1)}{3}$.

7. (i.) 392 ; (ii.) 3640.

8. 4285.

9. $\dfrac{1}{2}\left\{\dfrac{1}{2}-\dfrac{(-1)^n}{n^2+3n+2}\right\}$; $\dfrac{1}{4}$.

10. $(2n-1)2^{n+1}+2$.

11. $\dfrac{n(n+1)(n+4)(n+5)}{4}$; 63000.

12. $1\frac{1}{13}$; 2.

13. (i.) 120 ; (ii.) 1330.

14. 51 : 140.

15. 29799.

16. 48048.

18. $\dfrac{n}{5n+1}$; $\dfrac{1}{5}$.

19. $\dfrac{n(n+1)}{2}$.

22. 4.

23. $4n^2-6n+3$; 4285.

24. $19x^2+67x^3+239x^4+851x^5$.

25. $\dfrac{T_1(1-a)+T_2-bT_{n-1}-(a+b)T_n}{1-a-b}$.

26. $(2n^3-10n^2+19n-10)x^{n-1}$.

EXAMPLES. XIV.

(1.) (α.) $x=1$; $y=3$. (β.) $x=2$, 15, 28, 41, 54 ; $y=17$, 13, 9, 5, 1. (γ.) $x=12$, $y=9$.

2. (i.) 8 ; (ii.) 1.

3. (i.) $x=7+11t, y=3+17t$; 7, 3. (ii.) $x=7+19t, y=5+15t$; 7, 5.

4. 21.

5. 5 cows, 3 horses.

6. 24 half-crowns, 2 shillings, and 54 fourpenny bits.

7. (α.) $x=2, 8, 17, 5, 11, 3, 1$; $y=11, 3, 1, 5, 2, 8, 17$.
(β.) $x=2, 3, 6, 11$; $y=12, 7, 4, 3$.

8. 102, 10 ; 59, 45 ; or 16, 80.

9. (i.) 50 ; (ii.) 12.

10. 6.

EXAMPLES. XV.

1. $x^3-8x^2+19x-15=0$.

2. $x^3-\frac{9x^2}{2}+\frac{13x}{2}-\frac{15}{4}=0$.

3. 0, $2p$, $3r$.

6. $x^3-px^2+(4q-p^2)x+p^3-4pq+8r=0$; $-\frac{1}{2}$, $\pm\frac{3}{4}$.

7. $a=-1$.

9. -3, 7, 9.

10. $-1-\sqrt{-1}$, $-1\pm\sqrt{2}$.

11. $-\frac{3}{2}$, $-\frac{2}{3}$, $\frac{1}{6}$.

12. $\frac{1}{4}$, $\frac{1}{2}$, 1.

13. $x^4-2x^2+9=0$.

14. $\frac{1-\sqrt{-3}}{2}$, 3, $-\frac{2}{3}$.

15. $2-\sqrt{-3}$, $-2\pm\sqrt{-1}$.

16. (i.) 1, 3, 4 ; (ii.) 1, 3, $-2\pm\sqrt{-3}$.

17. $x^5+11x^4+42x^3+57x^2-13x-60=0$.

18. *1, 3, 9.*

19. $\pm\frac{1}{2}$, 8.

20. 2, $\frac{1}{2}$, 1.

22. (i.) 65231 ; (ii.) -258.

23. (i.) $3x^3+12x+41$, 172 ; (ii.) $2x^3-x^2+3x-7$, 12 ;
(iii.) $5x^2-34x+238$, -1665.

24. (i.) $5x^3+22x^2+74x+220$, 650 ; (ii.) $5x^3+32x^2+168x+838$, 4180 ;
(iii.) $5x^3-3x^2+14x-30$, 50.

25. -2, $+1$, $+1$.

26. 1207 ; 763.

27. $2x^4+(8h+1)x^3+(18h^3+3h^2+6h)x+(2h^4+h^3+3h^2+2)=0$,
where $h\cdot\frac{\pm\sqrt{15}-1}{8}$.

28. 2.

29. 6.

31. $x^2-35x+216=0$.

32. $\frac{3\pm\sqrt{5}}{2}$, $\frac{1\pm\sqrt{-3}}{2}$.

33. $\frac{1}{2}$, 4, $-\frac{2}{3}$.

34. ± 3, 1, -2.

35. $x^4-2x^3+(b-a^3+1)x^2-2bx+b(1-a^3)=0$.

36. $1+\sqrt{5}$, -1, -2.

37. $+2$, $+2$, -1.

38. 2, 3, 6.

EXAMPLES. XVI.

1. $0x^4-6a^5x^3+2a^6x^2$.

4. $\frac{2a}{\sqrt{x+a}}$.

5. $(1+x)(1-x)(1+y+x-xy)(1+y-x+xy)$.

6. $+\frac{3}{2}$ and $-\frac{7}{2}$.

11. 15.

12. (i.) $\frac{9a^2}{4}$; (ii.) 3; (iii.) 1; (iv.) 5; (v.) -1; (vi.) $-\frac{1}{2}$; (vii.) $\frac{a}{b}$;
(viii.) $\frac{7}{5}, \frac{4}{3}$.

14. $x^2+y^2+z^2+yz+zx+xy$.

21. $(x+1)(x-3)(x-7)$.

24. $-\frac{4}{3}$.

25. $y^3+(2q-p^2+3)y^2+(4q+q^2+3-2p^2-2pr)y+(q+1)^2-(p+r)^2=0$.

29. $\frac{a^2}{9}+\frac{b^4}{25}+\frac{c^6}{49}+\frac{d^8}{81}-\frac{2ab^2}{15}+\frac{2ac^3}{21}-\frac{2ad^4}{27}-\frac{2b^2c^3}{35}+\frac{2b^2d^4}{45}-\frac{2c^3d^4}{63}$.

30. $x^{\frac{5}{3}}-x^{\frac{4}{3}}y^{\frac{2}{3}}+xy^{\frac{4}{3}}-x^{\frac{2}{3}}y^2+x^{\frac{1}{3}}y^{\frac{8}{3}}-y^{\frac{10}{3}}$.

31. 12, 16; or 48, 4.

32. $p+1:q+1:r+1$.

34. 3.

35. $\frac{5}{x+1}-\frac{4}{x+2}-\frac{1}{x}:1$.

38. $\frac{1}{27}x^{\frac{3}{2}}-\frac{1}{8}y^9$.

40. -1.

42. 2.

44. $a+b+c$.

45. $\frac{a\sqrt{a+b+c}}{c\sqrt{b+c-a}}$.

46. 8 : 5.

47. $(ax-by)(bx-ay)$; $(b+c-1)(b^2+c^2+1-bc+b+c)$;
$(x^2+\sqrt{2}xy+y^2-1)(x^2-\sqrt{2}xy+y^2-1)$.

48. 7.

49. $(x+2y+3z)^2+(4y+5z)^2+(6z)^2$.

50. $\frac{\sqrt{2y}+\sqrt{x+y}+\sqrt{x-y}}{\sqrt{2}}$.

54. 3, 4, 6.

55. 10·00999.

57. $9x^2+2x-9$.

59. $(a+b+c+abc)(b-c)(c-a)(a-b)$.

60. $\dfrac{15x^2-52x+77}{(5-3x)^3}$; $\dfrac{6x^2}{(1-3x-5x^2)}$.

61. $\dfrac{18x^2+x}{15x^2-19x+6}$.

62. $x-3$.

63. (i.) -5, 13; (ii.) 7; (iii.) $-\dfrac{3}{2}\pm\dfrac{1}{2}\sqrt{\dfrac{-63}{11}}$, 3.

64. ± 609; 121.

65. $16\frac{4}{11}$ minutes past 12 o'clock.

66. 343 : 540.

67. 36.

68. 40, 23.

70. $x=\dfrac{a}{(a+b)^{\frac{1}{3}}}, \dfrac{b}{(a+b)^{\frac{2}{3}}}; y=\dfrac{b}{(a+b)^{\frac{2}{3}}}, \dfrac{a}{(a+b)^{\frac{1}{3}}}$.

71. $r^2x^3-prx^2+qx-1=0$.

72. $6\frac{2}{3}$.

73. $\dfrac{n(n+1)(n+2)}{2}$.

74. $x-5$.

75. $\dfrac{16x^2-5x-13}{(2+3x)^3}$; $\dfrac{2x(1+x)}{1-x^2-2x^3-x^4}$.

76. $\dfrac{x^2}{28x^2-5x-12}$.

77. (i.) 14; (ii.) 13; (iii.) $x=2, 4 : y=4, 2$.

78. $\pm 3\frac{3}{4}$, 169.

79. 288, 244.

80. 24.

81. £3, 5s.

83. (i.) x^3; (ii.) $\frac{a^3}{b^3}$.

84. $a - \frac{1}{a}$.

86. x must be $> \frac{3}{2}$ or < -2.

87. £40

89. $\frac{1}{3}$.

90. $\frac{25921}{567}$.

92. $(x-3)(2x-1)^2(x^2-x+4)$.

94. $\sqrt{10} + \sqrt{7}$ is the greater.

95. (i.) 2; (ii.) $-\frac{2ab}{a+b}$.

96. 30; $(\underline{|5})^6$.

97. 5s.; 4d.

98. 512.

99. $4x^2 + 7y^2 + 9z^2 + 3yz - 6zx + 8xy$.

100. $\frac{x^2(5x-1)}{2x^3-1}$; (i.) 0; (ii.) 1.

102. $1\frac{2}{13}$.

103. (i.) 3; (ii.) $\frac{2(a^2+ab+b^2)}{3(a+b)}$; (iii.) $\pm\frac{2a\sqrt{b}}{1+b}$.

104. $(y+z)(z+x)(x+y)$.

105. $\frac{4(a^2-c^2)}{ac}$.

106. A's stock $=$ £600 or £$22\frac{2}{9}$, B's $=$ £400 or £$111\frac{1}{9}$; A's gain $=$ £60 or £$16\frac{2}{3}$, B's $=$ £40 or £$83\frac{1}{3}$.

107. 126.

108. (i.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$; (ii.) $4\frac{11}{16}$.

109. 20, 25.

110. $\frac{1853\sqrt{3}}{90}$; $17 + 12\sqrt{2}$; $2\sqrt{5} - 3\sqrt{2}$.

111. (i.) $9\frac{473}{648}$; (ii.) $-4\frac{1}{2}$.

112. 11.

113. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4.$

114. $\frac{a + (b - ar)x - (lr + ks)x^n - ksx^{n+1}}{1 - rx - sx^2}.$

115. (i.) 3, 5, or 5, 3 ; (ii.) ± 4, ± 5.

116. 6 miles an hour.

117. (i.) $\frac{x - 3x^2 + 3x^3}{(1 - x)^3}$; (ii.) $\frac{x^2 + a^2}{x - a}.$

118. $2x^2 + 2xy - y^2.$

119. 522256, remainder 1 ; *t4tee* ; $25\cdot\dot{1}$.

120. $330x^7$; $\frac{7 \ldots (2r+3)}{1 \cdot 2 \ldots (r-1)}x^{r-1}$; $1 + \frac{3}{2}x^2 + \frac{15}{8}x^4 + \frac{35}{16}x^6 + \frac{375}{128}x^8.$

122. 240 sovereigns, 720 shillings, and 1200 pence.

124. 12 persons ; £14, 18s.

125. $\frac{5}{36}$; 221030 to 1225.

126. $+ x^{\frac{2}{2n+1}}y^{\frac{2n-2}{2n+1}} - x^{\frac{1}{2n+1}}y^{\frac{2n-1}{2n+1}} + y^{\frac{2n}{2n+1}}$; $2n + 1$.

127. (i.) $\frac{ab}{2c}$, $2d$; (ii.) 7, 2, 4 or $-\frac{29}{8}, -\frac{13}{2}, \frac{49}{8}.$

128. 58 sovereigns and 42 half-crowns.

130. $\frac{x^2}{2} + ax + \frac{a^2}{3}.$

131. 2,

132. $a^{\frac{1}{4}} - b^{\frac{1}{4}}.$

133. 14 feet and 32 feet.

134. 12 hours. 1 hour 30′ A.M.

135. 0.

136. 3, 9, 27 : 48, -36, 27.

137. $3a^2 - ab + 5b^2.$

138. $\frac{9q^2}{8p}.$

139. (i.) $\dfrac{1}{x+y}$; (ii.) $x^{\frac{5}{12}}y^{\frac{11}{12}}$.

141. $y^2-2y^4+3y^6-4y^8$.

142. $1-\dfrac{2y}{c}+\dfrac{5y^2}{c^2}-\dfrac{12y^3}{c^3}$.

143. a, b.

144. $n-1, \dfrac{n^2}{n+1}, \dfrac{n^3-n^2+n-1}{n^2}$.

145. $\dfrac{7(a^2+x^2)}{b^2+y^2}$.

146. $m+1$.

147. $q^2x^2-(p^3-3pq)x+q=0$.

148. $4n^2d+2n(a+d)+a$.

149. ab.

150. 90.

151. (i.) 7; (ii.) a, $b-a$.

152. $1-5x-2x^2$.

154. $\dfrac{(n-1)x^n-nx^{n-1}+1}{n(x-1)^2x^{n-1}}$.

155. 30 miles.

162. $\frac{3}{140}$.

164. $3+\dfrac{1}{6}\,\dfrac{1}{+6+6+\text{etc.}}$: 3, $\frac{19}{6}$, $\frac{117}{37}$, $\frac{721}{228}$, $\frac{4443}{1405}$.

165. $-\frac{1}{2}$.

166. $z^3-45z-100=0$; $x=z+4$.

167. (i.) $\sqrt{2}+\sqrt[3]{4}$; (ii.) $\sqrt[3]{9}-\sqrt[3]{3}$.

168. $z^3+6z-2=0$; $x=z+2$.

169. -1, 2, -4.

170. $3\frac{1}{4}$ miles and 3 miles an hour.

171. 1147.

172. 56 and 44.

173. 3, 7, 11, 20.

174. (i.) $\frac{3}{4}$; (ii.) $\frac{3}{7}$.

175. 42925.

176. 16575.

177. $\dfrac{2n^2+n-1}{4n(2n+1)}$; $\dfrac{1}{4}$.

178. 6.

179. 3, 4, 6.

180. 9.

181. $\frac{5}{188}$.

182. $1+2x-7x^2-16x^3$.

183. (i.) 100; (ii.) ± 3, ± 4, ± 6.

184. (i.) Yes; (ii.) No; (iii.) No.

185. $7, \ -1, \ -3 \pm \sqrt{-34}$.

189. (i.) $\dfrac{9}{4} - \dfrac{2n^2+8n+9}{4 \cdot 3^n}$; (ii.) $\dfrac{n(n+1)(n+2)}{18}$.

191. $x-2a$; $2a$.

192. $a=c^3-3cd$; $b=d^3$.

194. $r+1$; $\dfrac{1 \cdot 3 \cdot 5 \ldots (2r-1)}{\lfloor r \, 2^r}$.

195. 4 gallons and 3 gallons.

196. (i.) $\dfrac{1}{n(n+5)}$; $\dfrac{137}{300} - \dfrac{1}{5}\left\{\dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \dfrac{1}{n+4} + \dfrac{1}{n+5}\right\}$;

(ii.) $8\left(-\dfrac{3}{2}\right)^{n-1}$; $\dfrac{16}{5}\left\{1 - \left(-\dfrac{3}{2}\right)^n\right\}$.

197. $\dfrac{2-5x}{(1-x)(1-2x)}$.

200. $a_n = \dfrac{1}{2}\left(a+b+\dfrac{a-b}{3^n}\right)$, $b_n = \dfrac{1}{2}\left(a+b-\dfrac{a-b}{3^n}\right)$.

201. $8 - \sqrt{15}$.

202. 5.

203. $2x^3$ is $>$ or $<$ $x+1$ according as x is $>$ or <1.

204. $0, \ 2, \ -3$.

205. 16, 2, 130.

206. $a = n^2 - n + 1$.

207. $\frac{441}{1000}$, $\frac{21}{40}$.

208. $3x^2 + 29x + 126$; 490.

209. $x^3 - 8x^2 + 19x - 15 = 0$.

210. 1125 feet per second, and 500 feet per second.

211. $\dfrac{715}{2^{16}}$.

212. $\dfrac{\sqrt{m}(m+3n)}{4}$.

213. ± 2, ± 3, ± 4.

214. (i.) 819 ; (ii.) 455.

215. $3x^4 - 9x^3 + 20x^2 - 52x + 146$; -385.

216. 5100.

217. $\dfrac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$.

219. (i.) $-a, b$; (ii.) $-\dfrac{a^2}{b}, -\dfrac{b^2}{a}$.

221. $(ax + by + cz)(lx + my)$.

222. $-\dfrac{p_1}{np_0}$.

223. 0.

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